UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 1 Examination in Engineering: May, 2022

Module Number: IS1402 **Module Name: Mathematical Fundamentals for Engineers**

[Three hours]

[Answer **all questions,** each question carries twelve marks]

Q1. a) Given that

$$z = \frac{\sqrt{-26 - 6\sqrt{3}i}}{2 + \sqrt{3}i}$$

i.) Express z in the form $a + bi$, where $a, b \in R$ and $a > 0$.

ii.) Convert z into polar form.

iii.) Write down the modulus and the argument of z.

[3 Marks]

b) Let n be a positive integer, and

$$w_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, 2, \ldots, n-1$$

Show that

i.) w_k is a one of the n^{th} roots of unity.

ii.) If w is the primitive root of unity then $w_k = w^k$

iii.) For $n > 1$, summation of all the n^{th} roots of unity is zero.

[4 Marks]

c) If ω is a complex cubic root of unity, without computing ω, show that

$$\begin{bmatrix} \omega & \omega^2 & -1 \\ \omega^2 & -1 & \omega \\ -1 & \omega & \omega^2 \end{bmatrix}^2 = 2\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

[2 Marks]

d) Express $\cos^5\theta$ in terms of muliples of $\cos\theta$ and $\sin\theta$.
Hence, find

$$\int_0^{\pi/2} \cos^5\theta \, d\theta$$

[3 Mark]

Q2. a) i.) Briefly explain singular and non-singular matices by giving an example for each.

ii.) Write down five properties of the determinant of a matrix.

[4 Marks]

b) Define

i.) a minor

ii.) a cofactor

iii.) the adjoint

of a matrix.

Find the adjoint of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

[4 Marks]

c) Determine the values of α and β, which the system

$$\begin{aligned} 2x + y + z &= 3 \\ x - 2y + 3z &= \alpha \\ x + 3y + \beta z &= 4 \end{aligned}$$

has

i.) unique solution

ii.) infinitely many solutions

iii.) no solutions

Solve the system for $\alpha = -6$ and $\beta = 3$.

[4 Marks]

Q3 a) i.) Explain what is meant by the statement, '$f(x)$ is continuous at a point a'.

ii.) Discuss the continuity of the function

$$f(x) = \begin{cases} 2x - 1 & x < -2 \\ x + 1 & -2 \leq x < 1 \\ 5 - 3x & x \geq 1 \end{cases}$$

iii.) Sketch the graph of $y = |2x - 1| - 2|x + 1| + |x - 2|$

[5 Marks]

b) i.) Show that $f(x) = |x - 2|$ continous but not differentiable at $x = 2$.

ii.) If f and g are differentiable functions show that $f \circ g$ is also differentiable.

iii.) Let $f(x) = x^3$ and $g(x) = \tan x$. Determine $(f \circ g)'$ by using ii.) in above.

[5 Marks]

c) Evaluate the following limits, if they exist.

i.) $\lim_{x\to 0} \frac{1-e^{1/x}}{2+e^{1/x}}$ ii.) $\lim_{x\to 1} \frac{(x^2+1)(x^2+x-1)+x^2-3}{x-1}$

[2 Marks]

Q4. a) i.) State the Mean Value Theorem.

ii.) Let f and f' be continuous and differentiable functions on an open interval $(\alpha,\ \beta)$, where $\alpha/2 < a < b < \beta/2$. If the second derivative of f is zero on (α, β) for all x in the given interval, by applying Mean Value Theorem twice in the intervals $[2a, a+b]$ and $[a+b, 2b]$, show that

$$f(2a) + f(2b) = 2f(a+b).$$

[3 Marks]

b) i.) If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then show that there exists $c \in (a, b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

ii.) State the L'Hospital's Rule

iii.) Use L'Hospital's Rule to evaluate

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$$

iv.) Obtain the power series expansion of $\log(1+x)$ about $x = 0$.

[5 Marks]

c) i.) Let $z = f(x, y)$ be a continuous and differentiable function on R^2. Show that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

ii.) If $z = f(x, y)$, where $x = e^r\cos\theta$ and $y = e^r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \neq e^{2r}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

[4 Marks]

Q5. a) i.) Briefly explain what is meant by 'Unit vector' and 'Position vector'.

ii.) OAB is a triangle such that $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{OB} = 3\mathbf{b}$. Find $\overrightarrow{AB}$ in terms of a and b.

If P is the point on AB such that $AP: PB = 2:3$, show that $\overrightarrow{OP}$ is parallel to the vector, $(\mathbf{a} + \mathbf{b})$.

[4 Marks]

b) A rigid body is spinning with angular velocity 9 radians/sec about an axis OR, where R is $(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and O is the origin. Find the velocity of the point $(-3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ on the body.

[3 Marks]

c) i.) The temperature at any point in space is given by $T = 2xy + yz + zx$. Determine the derivative of T, (i.e. ∇T), in the direction of the vector $(4\mathbf{i} - 3\mathbf{k})$ at the point (1,1,1).

ii.) Given the vector field,

$$\mathbf{V} = (x^2 - y^2 + 2xz)\mathbf{i} + (xz - xy + yz)\mathbf{j} + (x^2 + z^2)\mathbf{k}.$$

Find $curl(\mathbf{V})$.

Show that the vectors given by $curl(\mathbf{V})$ at $P_0(0, 2, -1)$ and $P_1(-2, -3, 13)$ are orthogonal.

[5 Marks]