University of Ruhuna

Bachelor of Science (General) Degree Level I (Semester II) Examination

January/February 2022

Subject: Applied Mathematics/Industrial Mathematics
Course Unit: AMT122β/IMT122β(Mathematical Modelling I)

Time: Two (02) Hours

Answer <u>all</u> the questions.

Calculators provided by the university are allowed

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1. a) Find all equilibria of each of the following differential equations, classify their stability and sketch phase line diagrams.

(i) $\dfrac{dy}{dx} = -15 + 8y - y^2$,

(ii) $\dfrac{dy}{dx} = y(y^2 - 2y + 1)$.

[40 Marks]

b) Let $U(n)$ and $V(n)$ be the total amount of pollutants in lakes A and B respectively. 30% of the pollutants from lake A and 10% of the pollutants from lake B are removed every year. Also, the pollutant that is removed from lake A is added to lake B due to the flow of water from lake A and B, also it is assumed that 3 tons of pollutants are directly added to lake A and 10 tons of pollutant are added to lake B.

(i) Construct a discrete dynamical model for the above scenario. *[20 Marks]*

(ii) Find the equilibrium point of the model and determine its stability. *[20 Marks]*

(iii) It is determined that an equilibrium level of a total of 8 tons of pollutants in lake A and a total of 60 tons in lake B would be acceptable. What restriction should be placed upon the total amount of pollutants that are added directly, so that these equilibria can be achieved? *[20 Marks]*

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2. a) A tank initially contains $400l$ of water. A solution containing $2g/l$ of chemical flows into the tank at a rate of $5l/min$, and the mixture flows out at a rate of $3l/min$. Let $Q(t)$, be the amount of chemical in the tank at time t.

(i) Show that $Q(t)$ satisfies the differential equation
$$\frac{dQ(t)}{dt} + \frac{3Q(t)}{400+2t} - 10 = 0.$$
[20 Marks]

(ii) Obtain an expression for $Q(t)$. *[20 Marks]*

(iii) If the volume of the tank is $900l$, how long will it take to start the tank overflow. *[10 Marks]*

Continued.

(iv) Find the amount of the chemical in the tank at the moment of tank starts to overflow. *[10 Marks]*

b) An ice-cream cup of temperature $-20°C$ is taken out from a freezer. The outside temperature is $30°C$. After five minutes, the ice-cream cup has warmed to $-10°C$.

(i) Formulate a **continuous** mathematical model to determine the temperature, $T(t)$, of the ice-cream cup at any time t after it is taken out from the freezer. *[15 Marks]*

(ii) Find an expression for $T(t)$ in terms of t. *[20 Marks]*

(iii) What is the temperature of the ice-cream cup after ten minutes it is taken out from the freezer? *[05 Marks]*

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3. a) A visitor who was infected with flue virus entered an isolated island with a 999 total population. After 5 days, 10 people are infected. Assume the rate of spreading of the virus is proportional to the number infected and to the number uninfected. Let $I(t)$ be the number of infected individuals at time t. Assume that $I(t)$ satisfies the differential equation $\frac{dI(t)}{dt} = kI(t)(1000 - I(t))$, where k is a positive constant.

(i) Find the equilibria of the above differential equation model. *[10 Marks]*

(ii) Describe the stability of the equilibria. *[10 Marks]*

(iii) Draw a phase line diagram to interpret the stability behavior of the equilibrium points *[10 Marks]*

(iv) Obtain an expression for $I(t)$ in terms of t and k. *[20 Marks]*

(v) Obtain an expression to determine the value of k. *[10 Marks]*

(vi) What will happen to $I(t)$ as time tends to infinity. *[10 Marks]*

b) Consider discrete dynamical model $U_{n+1} = -0.4U_n^2 + 3.4U_n - 2$.

(i) Find the critical points of the above model. *[10 Marks]*

(ii) Describe the stability of the critical points by using relevant calculations. *[10 Marks]*

(iii) Draw a Cob-Web diagram near each critical point, and confirm your result in part b(ii). *[10 Marks*

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4. Consider the following non-linear system of differential equations:

$$\frac{dx}{dt} = -4x + y^2 + 4,$$
$$\frac{dy}{dt} = (x - 2)y.$$

a) Find the critical points of the above system. *[20 Marks]*

b) If (x_e, y_e) is a critical point of the above system. Write down the Jacobian matrix of the system at (x_e, y_e). *[10 Marks]*

c) Determine the stability types of each of the critical points. *[Marks 30]*

d) Find the straight-line trajectories about the critical points if exist. *[20 Marks]*

e) Sketch the phase plane diagram of the above system by determining the behavior of the trajectories near equilibrium points. *[20 Marks]*
