UNIVERSITY OF RUHUNA
FACULTY OF SCIENCE
Bachelor of Science (General) Degree
Level I (Semester II) Examination – January, 2022

SUBJECT: MATHEMATICS **COURSE UNIT: MAT 121β – ALGEBRA**

INSTRUCTIONS:

- Answer **ALL** Questions.
- Time Allowed: **TWO (02)** hours.

❖ Here Z, Z^+ and $\boldsymbol{R}$ denotes the set of integers, the set of positive integers, and the set of real numbers respectively.

1. (a) Let $\{A_n\}_{n=1}^{\infty}$ be an infinite sequence of subsets of a universal set U.

(i) Show that

$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) \subseteq \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right).$$

[20 marks]

(ii) It is further given that $A_{n+1} \subset A_n$ for all $n \in Z^+$.
Show that

$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right).$$

[15 marks]

(iii) Now, let $A_n = \left(-\frac{1}{n},\ 1+\frac{1}{n}\right)$ for all $n \in Z^+$.
Using the fact that $A_{n+1} \subset A_n$ for all $n \in Z^+$, find

$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right).$$

[10 marks]

(b) Let R be a relation on Z defined by mRn if $m+2n=3k$ for some $k \in Z$.

(i) Show that R is an equivalence relation. [15 marks]

(ii) Find [1], the equivalence class of 1. [10 marks]

(c) Let $f(x)=\sqrt{x-1}$ and $g(x)=\dfrac{1}{\sqrt{x-1}}$.

(i) Write down the domain of g. [05 marks]

(ii) Find $g \circ f$. [15 marks]

(iii) Find the domain of $g \circ f$. [10 marks]

Ctd...

2. (a) Let A and B be invertible matrices of order 4 with $\det(A) = \alpha$ and $\det(B) = \beta$.

Compute (i) $\det(\beta A)$ (ii) $\det(AB)$ (iii) $\det(B^{-1})$ (iv) $\det(A^T B^{-1})$

[20 marks]

(b) Let a, b and c be non-zero real numbers.

Using the properties of determinants, show that

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{pmatrix} = abc - ab - bc - ca.$$

[20 marks]

Hence, find

$$\det\begin{pmatrix} \alpha\beta & \alpha & \alpha & \alpha \\ \beta & 0 & 0 & 1 \\ \beta & 0 & 1 & 0 \\ \beta & 1 & 0 & 0 \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbf{R}.$$

[15 marks]

(c) Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 7 & 1 \\ 3 & 5 & 2 \end{pmatrix}$.

(i) Find $\det(A)$ and $\operatorname{adj}(A)$.

[20 marks]

(ii) Show that $A^{-1} = \frac{1}{3}\begin{pmatrix} 9 & -1 & -4 \\ -3 & 1 & 1 \\ -6 & -1 & 5. \end{pmatrix}$.

[10 marks]

(iii) Solve the following system of linear equations:

$$\begin{aligned} 2x &+ 3y + z = 3 \\ 3x &+ 7y + z = 3 \\ 3x &+ 5y + 2z = 6 \end{aligned}$$

[15 marks]

Ctd...

3 (a) (i) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with integer coefficients. If $\frac{r}{s}$, where $(r, s) = 1$ is a rational root of $f(x)$, show that $r|a_0$ and $s|a_n$. [20 marks]

(ii) Let $f(x) = x^5 - 5x^4 + 9x^3 - 7x^2 + 2$.

Find the rational root of $f(x)$. [10 marks]

It is given that $1 + i$ is a root of $f(x)$. Show that $x^2 - 2x + 2$ is a factor of $f(x)$. [15 marks]

Find the other roots of $f(x)$. [10 marks]

(b) **Use Newton's identities** to find

$$(1+i)^5 + (1-i)^5.$$ [20 marks]

(c) Let $p(x) = x^3 - 4x^2 + 5x - c$, where $c \in R$ and let α, β and γ be the roots of $p(x)$.

Write down *Vieta's formulas.* [10 marks]

If $\beta = \gamma$, show that there are two such polynomials. [15 marks]

4 (a) (i) Obtain the cube roots of unity in the form $a + ib$, where $a, b \in R$. [20 marks]

(ii) Solve $z^3 = 27$. [15 marks]

(iii) Let ω be a complex cube root of unity. Simplify $(\omega^3 + 1)(\omega^2 + 1)(\omega + 1)$. [15 marks]

(b) Let $\oplus$ be a new operation of addition on Z defined by $m \oplus n = m + n - 1$ and let $\otimes$ be a new operation of multiplication on Z defined by $m \otimes n = m + n - mn$.

Show that $(Z, \oplus, \otimes)$ is ring. [50 marks]