University of Ruhuna

Bachelor of Science General Degree
Level II (Semester II) Examination - June 2022

Subject: Mathematics
Course Unit: **MAT222δ (Real Analysis II)**

Time :One (01) Hour

Answer ALL Questions.

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1. a) For a functional sequence $\{f_n(x)\}_{n=1}^{\infty}$ on $D \subset \mathbb{R}$, in the usual notation, **define** the pointwise convergence and the uniform convergence.

   Find the limit functions of the followings:

   (i) $f_n(x) = e^{-nx}$; on $x \in [0, \infty)$,

   (ii) $f_n(x) = \dfrac{1 - \sin^2 nx}{n^2}$; $x \in \mathbb{R}$.

   (**40** Marks)

   b) Let $\{f_n(x)\}_{n=1}^{\infty}$ be a functional sequence such that

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ for } x \in [a, b], \text{ and}$$

$$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

   Show that $f_n(x) \to f(x)$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.
   Hence, determine the uniform convergence of $f_n(x) = x^2e^{-nx}$, where $x \in (0, \infty)$.

   (**30** Marks)

   c) Consider the functional sequence $\{f_n(x)\}_{n=1}^{\infty}$,

$$f_n(x) = \begin{cases} n^2x, & 0 \le x \le 2/n \\ -n^2x + 4n, & 2/n \le x \le 4/n \\ 0, & 4/n \le x \le 1. \end{cases}$$

   (i) Sketch the graph of $f_n(x)$ clearly labelling the axes.

   (ii) Find $\int_0^1 f_n(x)dx$.

   (iii) Check the uniformly convergence of $f_n(x)$ to $f(x)$ on [0,1], where $f(x)$ is limit function of $f_n(x)$. Further, clearly state the theorems you may use.

   (**30** Marks)

2. a) In the usual notation, state the Weierstrass M test for functional series.

Hence, determine the uniform convergence of the following functional series,

$$\frac{x}{1+1^2.x^2}+\frac{x}{2+2^2.x^2}+\frac{x}{3+3^2.x^2}+\dots$$

on [a,b] where $a < b$ and $a, b \in \mathbb{R}$.

(30 Marks)

b) In the usual notation, state the Dirichlet test for a functional series.

Hence, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2},$$

is uniformly convergent for all $x \in \mathbb{R}$.

(30 Marks)

c) Consider the series of function:

$$\sum_{n=1}^{\infty} \frac{\sin(nx^2+x)}{n(n+1)}, \; x \in \mathbb{R}.$$

Check whether the above series of function is uniformly convergent?

(40 Marks)

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