University of Ruhuna

Bachelor of Science General Degree
Level II (Semester II) Examination - June 2022

Subject: Mathematics
Course unit: MAT225β (Mathematical Statistics-I)

Time: Two (02) Hours

Answer all questions.

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1. a) Let Y_1 and Y_2 be jointly distributed random variables with means μ_1 and μ_2 respectively. In the usual notation define the covariance of Y_1 and Y_2.
Show that $Cov(Y_1, Y_2) = E(Y_1Y_2) - \mu_1\mu_2$.

(20 marks)

b) The joint probability distribution of discrete random variables Y_1 and Y_2 is given by the following table.

y_1 \ y_2	0	1	2	3
0	1/12	1/4	1/8	1/120
1	1/6	1/4	1/20	0
2	1/24	1/40	0	0

(i) Find $P(Y_1 = 0,\ 1 \leq Y_2 < 3)$
(ii) Find $P(Y_1 > Y_2)$
(iii) Calculate $E(Y_1Y_2)$, $E(Y_1)$ and $E(Y_2)$.
(iv) Hence find, the $Cov(Y_1, Y_2)$.
(v) Determine whether Y_1 and Y_2 are independent or not. Justify your answer.

(80 marks)

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2. a) The joint probability density function of random variables Y_1 and Y_2 is given by

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} \frac{2}{3}(y_1 + 2y_2) & \text{, if } 0 < y_1 < 1 \text{ and } 0 < y_2 < 1 \\ 0 & \text{, otherwise} \end{cases}$$

Find

(i) the marginal probability density function of Y_2.
(ii) conditional density function of Y_1 given $Y_2 = \frac{1}{2}$.
(iii) conditional mean of Y_1 given $Y_2 = \frac{1}{2}$.
(iv) conditional variance of Y_1 given $Y_2 = \frac{1}{2}$.

(100 marks)

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3. a) If the probability density function of the random variable Y is given by

$$f_Y(y) = \begin{cases} 6y(1-y) & , \text{if } 0 < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

find the probability density function of $U = Y^3$.

(40 marks)

b) Y_1 and Y_2 are standard normally distributed independent random variables. If $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$, find the joint probability density function of U_1 and U_2.

(60 marks)

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4. a) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n taken from a normal distribution with mean μ and variance σ^2.

I Define the sample mean, $\bar{Y}$ and the sample variance, S^2.

(10 marks)

II Write down the distributions of the following random variables.

$$\sqrt{n}\left(\frac{\bar{Y}-\mu}{\sigma}\right), \quad \sum_{i=1}^{n}\left(\frac{Y_i-\mu}{\sigma}\right)^2, \quad \sum_{i=1}^{n}\left(\frac{Y_i-\bar{Y}}{\sigma}\right)^2$$

Let $\mu = 0$, $\sigma^2 = 4$ and $n = 16$

(i) If $P(|\bar{Y}| < a) = 0.5$, find a.
(ii) If $P(\alpha_1 < S^2 < \alpha_2) = 0.9$ and $P(S^2 < \alpha_1) = P(S^2 > \alpha_2)$, find α_1 and α_2.

(50 marks)

b) State the central limit theorem.
The service times for customers coming through a checkout counter in a retail store are independent random variables with mean 1.5 minutes and variance 1.0 minutes^2. Calculate approximately the probability that 100 customers can be served in less than 2 hours of total service time.

(40 marks)
