UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 1 Examination in Engineering: October, 2022

Module Number: IS1402 **Module Name: Mathematical Fundamentals for Engineers**

[Three hours]

[Answer **all questions,** each question carries twelve marks]

PART A

Q1. a) Let $u = -1 + \sqrt{3}i$ and $w = 1 + \sqrt{3}i$.

i.) Express u and w in polar form.

ii.) Find square roots of u and w in the form of $a + ib$, where a and b are constants.

iii.) Solve the equation $z^4 - \sqrt{3}iz^2 - 1 = 0$

[4 Marks]

b) i.) Prove that for any integer $n > 0$, if

$$(\cos\theta + i\sin\theta).(\cos 2\theta + i\sin 2\theta). \dots .(\cos n\theta + i\sin n\theta) = 1,$$

then the general value of θ is $\frac{4k\pi}{n(n+1)}$; where $k \in \mathbb{Z}$

ii.) Let α and β be two roots of the equation, $x^2 - 2x + 4 = 0$.

Show that $\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$, where $n \in \mathbb{Z}$.

[4 Marks]

c) Show that $\sin i\theta = i\sinh\theta$

Hence, show that $$\sin\left(i\ln\left(\frac{a - ib}{a + ib}\right)\right) = \frac{2ab}{a^2 + b^2}.$$

[4 Marks]

Q2. a) i.) Define Symmetric and Skew-symmetri matrices. Give an example for each.

ii.) Show that all the diagonal elements of a skew symmetrc matix are zero.

iii.) Given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \neq 0$.

If $AA^t = A^tA$, show that either A is symmetric or $A - aI$ is skew symmetric. Where I is 2×2 unit matrix.

[5 Marks]

b) i.) Find α and β such that the matrix $A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & \alpha \\ 1 & \beta & 2 \end{bmatrix}$ is orthogonal.

ii.) Find λ and μ such that the matrix $B = \begin{bmatrix} 2 & -1 & -2 \\ 1 & \lambda & -2 \\ 1 & -1 & \mu \end{bmatrix}$ is involuntary.

[3 Marks]

c) Consider the system of linear equations

$$\begin{aligned} 2x + y - z &= 5 \\ x - 2y - 4z &= 4 \\ 3x + y + 2z &= 12 \end{aligned}$$

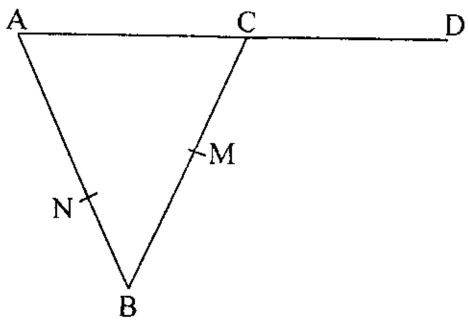
i.) Determine the ranks of coefficient and augmented matrices, and hence, show that the system has a unique solution.

ii.) Use the Cramer's rule to solve the system.

[4 Marks]

Q3. a) i.) Briefly explain what is meant by 'Position vector' and 'Equal vectors'

ii.) ABC is a triangle and N is a point on AB. If $\overrightarrow{AC} = a,\ \overrightarrow{AN} = 2b$ and $\overrightarrow{NB} = b$, find the vector $\overrightarrow{BC}$ in terms of a and b.



(Diagram is not in scale)

M is the midpoint of BC and C is the midpoint of AD. Show that N, M, D are lie along the same straight line.

[4 Marks]

b) A force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is applied at the point (1, -1, 2). Find the moment of the force about the point (2, -1, 3).

[3 Marks]

c) i.) Prove that $\nabla\emptyset$ is a vector perpendicular to the surface $\emptyset(x, y, z) = C$, where C is a constant.

ii.) Find the directional derivative of $\mathbf{V}^2$, where $\mathbf{V} = x^2y\mathbf{i} + y^2z\mathbf{j} + xz^2\mathbf{k}$, at the point $(1, 0, 2)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(2, 2, 1)$.

[5 Marks]

PART B

Q4. a) For any $a, b \in \mathbb{R}$, show that

i.) $|a+b| \leq |a| + |b|$

ii.) $||a| - |b|| \leq |a - b|$

iii.) $|ab| = |a||b|$

[3 Marks]

b) Sketch the graph of $y = |x-1| - 2|x+1| + x - 2$.

[2 Marks]

c) In each of the following i.) – iv.), write down an example by a real valued function f, which satisies the given conditions.

i.) f is one to one function, but not onto.

ii.) Limit of f does not exist for any $a \in \mathbb{R}$.

iii.) Limit of f is exists for all $x \in \mathbb{R}$, but f is not continous at every $a \in \mathbb{Z}$.

iv.) f is contionous for all $x \in \mathbb{R}$, but not differentiable at $x = 1$.

[4 Marks]

d) Evaluate the following limits

i.) $\lim_{x \to 1} \frac{\ln(3x-2)}{1-x^4}$ ii.) $\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\cot x - 1}$

iii.) $\lim_{x \to \infty} \frac{\sqrt{3x^2-2}}{4x+1}$

[3 Marks]

Q5. a) i.) State the Rolle's theorem.

ii.) Use Rolle's theorem to show that for any $k \in \mathbb{R}$, $4x^3 + 15x^2 + 12x + k = 0$ has at most one positive real root.

[3 Marks]

b) i.) State and prove the Mean Value Theorem.

ii.) If $0 < a < b < \frac{\pi}{2}$, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

Hence, deduce that $\frac{2}{5} < \tan^{-1}(2) < 2.$

[5 Marks]

c) i.) If z is a function of variables x and y, where $x = u\cosh\alpha - v\sinh\alpha$ and $y = u\sinh\alpha + v\cosh\alpha$; α being a constant, show that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

ii.) Write down the power series expansion of 2^x about $x = 0$.

[4 Marks]