UNIVERSITY OF RUHUNA
FACULTY OF SCIENCE

BACHELOR OF SCIENCE GENERAL DEGREE
LEVEL II (SEMESTER I) - END SEMESTER EXAMINATION - JANUARY, 2022

Subject: Mathematics **Course Unit: MAT 211β - Linear Algebra**

Time Allowed: 2 Hours **Answer ALL Questions**

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1. Let $A(\alpha) = \begin{bmatrix} 1 & 0 & \alpha \\ -\alpha & 1 & -\frac{\alpha^2}{2} \\ 0 & 0 & 1 \end{bmatrix}$; where $\alpha \in \mathbb{R}$.

(a) (i) Show that $A(\alpha)A(\beta) = A(\alpha + \beta)$ for $\alpha, \beta \in \mathbb{R}$. [20 marks]

(ii) ***Deduce*** that $A(\alpha)A(-\alpha) = I_3$. [15 marks]

(iii) Using ***the Principle of Mathematical Induction*** or ***otherwise***, show that

$$(A(\alpha))^n = A(n\alpha), \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

[20 marks]

(iv) ***Hence***, find $[(A(\alpha))^n]^{-1}$, the inverse of $(A(\alpha))^n$, for $n \in \mathbb{Z}^+$. [15 marks]

(v) Let $B = \left[\begin{array}{c|c} A(-1) & I \\ \hline 0 & A(1) \end{array}\right]$ be a block matrix.

Show that $B^{-1} = \left[\begin{array}{c|c} A(1) & -I \\ \hline 0 & A(-1) \end{array}\right]$. [20 marks]

(b) Now, let $C = A(-1)$. Express C as LU, where L is a unit lower triangular matrix and U is an upper triangular matrix. [10 marks]

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2. (a) Let $\alpha, \beta \in \mathbb{R}$ and consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \alpha \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ \beta \end{bmatrix}.$$

Determine for which values of α and β this system has:

(i) a unique solution, [10 marks]

(ii) no solutions, [10 marks]

(iii) infinitely many solutions. [10 marks]

(iv) In case of (iii), solve the system using elementary row operations and express the solution in the form, $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, where t is a parameter and $\mathbf{p}$ and $\mathbf{v}$ are vectors to be determined.

[20 marks]

(b) Let V be the vector space $\mathbb{R}^3(\mathbb{R})$ with the Euclidean inner product. Apply the Gram Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (0,1,1)$, and $\mathbf{u}_3 = (0,0,1)$ into an orthonormal basis. [50 marks]

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ctd..

3. Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) (i) Show that $p(\lambda)$, the characteristic polynomial of the matrix A, is given by

$$p(\lambda) = (\lambda - 1)^2(\lambda - 2).$$

[20 marks]

(ii) Write down the eigenvalues of A. [05 marks]

(iii) Find the eigenvectors of A. [30 marks]

(b) (i) Write down a matrix P that diagonalizes A. [05 marks]

(ii) Compute $P^{-1}A^5P$. [15 marks]

(c) (i) Verify the Cayley-Hamilton theorem for the matrix A. [15 marks]

(ii) **Hence**, show that $A^{-1} = \frac{1}{2}\left(A^2 - 4A + 5I\right)$. [10 marks]

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4. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for $\mathbb{R}^3$, where $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (0, 1, 1)$, and $\mathbf{u}_3 = (0, 0, 1)$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation for which

$$T(\mathbf{u}_1) = (0, 0, 0), \quad T(\mathbf{u}_2) = (-1, 1, -1), \quad T(\mathbf{u}_3) = (0, 0, -1).$$

(a) (i) Find a formula for $T(x, y, z)$, for all $(x, y, z) \in \mathbb{R}^3$. [25 marks]

(ii) Verify that T is linear. [25 marks]

(b) Find a basis for $\mathrm{Ker}(T)$ and a basis for $\mathrm{Im}(T)$. [20+20 marks]

(c) Verify that $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim \mathbb{R}^3$. [10 marks]

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