University of Ruhuna

Bachelor of Science General Degree
Level III (Semester I) Examination - 2021

Subject: Applied Mathematics
Course Unit: AMT311β
(Numerical Methods with Applications) .

Time: Two (02) Hours

Answer All Questions
Calculators provided by the university are allowed to use.

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1. a) In the usual notation, define the matrix norms, $||A||_1$, $||A||_2$ and $||A||_\infty$, where A is a matrix of order n.
   Find $||A||_1$, $||A||_2$ and $||A||_\infty$ for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 4 \\ 0 & -2 & 4 \end{pmatrix}.$$

   Define the condition number $\kappa(A)$ for a non-singular matrix A.
   Calculate $\kappa(A)$ for the above matrix A.

   b) Determine whether each of the given matrices is in row echelon form

$$B = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

   (i) By recording the row operations you perform, determine reduced row echelon forms of B and C.

   (ii) Obtain solutions for the system of linear equations

$$\begin{aligned} 3x_1 + 3x_2 + 4x_3 &= 10 \\ 2x_1 + x_2 + 3x_3 &= 12 \\ x_1 + x_2 + 3x_3 &= 3. \end{aligned}$$

   by converting into row echelon form, and applying back substitution.

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2. a) Obtain solution for the following linear system by using Gauss Jordan method with partial pivoting

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 4x_1 + 3x_2 - 3x_3 &= 3 \\ 3x_1 + 5x_2 + 3x_3 &= 2. \end{aligned}$$

Continued...

b) In the usual notation, use the decomposition $A = L + D + U$ in order to write down the general formula of the Jacobi iterative method for the system $Ax = b$.

c) Solve the system of equations

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 3 \\ x_1 + 3x_2 + x_3 &= 1 \\ x_1 + 3x_2 + 7x_3 &= 2. \end{aligned}$$

upto the second iterative $x^{(2)}$ using Jacobi iterative method with initial approximation $x^0 = (0, 0, 0)^T$.

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3. a) (i) Consider the initial value problem

$$\frac{dy}{dt} = f(y, t) = t^2 y - t + 5, \quad y(0) = 5.$$

Show that the function satisfies the Lipschitz condition on $D = \{(t, y) | 0 \leq y \leq 3, 0 \leq t \leq 4\}$.

(ii) In the usual notation, obtain general form of the Picard iteration formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s) ds, \quad n \geq 0,$$

for the initial value problem

$$y' = f(x, y); \quad y(x_0) = y_0.$$

Hence, solve the following initial value problem

$$y' = 2(y + 1); \quad y(0) = 0.$$

for $y_2(x)$.

b) Consider

$$\frac{dy^2}{dx^2} + 5\frac{dy}{dx} - 3y - 5x = 0, \quad \text{with } y(0) = 0 \text{ and } y'(0) = 1.$$

By using Heun's method and taking step size as 0.2, find the approximation for $y(0.2)$ and x' at $t = 0.1$

c) In the usual notation, express the general form of the fourth order Runge Kutta method to solve the initial value problem,

$$y' = f(x, y); \quad y(x_0) = y_0,$$

with step size h.

Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2 + 5, \quad y(0) = 5$$

By using fourth order Runge Kutta scheme, obtain the approximation at $x = 0.1$ with step size 0.1.

Continued...

4. In order to solve the heat equation $u_{xx} = cu_t$ numerically for $t > 0$ and $0 \leq x \leq a$ subject to boundary conditions $u(0,t) = T_0, u(a,t) = T_a$ and the initial condition $u(x,0) = f(x), 0 < x < a$, we use the discretization $x_i = ih$, for $i = 1, 2, \ldots n$ and $t_j = jk$ for $j = 1, 2, \ldots n$ where $h = a/n$, and $k > 0$ are step sizes in x and t axes respectively.

In the usual notation, derive

$$u_{i,j+1} = ru_{i+1,j} + (1-2r)u_{i,j} + ru_{i-1,j},$$

where $r = k/ch^2$.

Consider the heat equation $u_{xx} = 2u_t \quad t > 0$ and $0 \leq x \leq 1$,

with boundary conditions

$$T_0 = 1, \quad T_a = 0, \quad t > 0,$$

and initial condition $u(x,0) = 0, \quad$ for $0 < x < 1$.

a) Find the approximated temperature at each of the interior mesh points for time steps $j = 0, 1, 2$ when $h = 0.2$ and $k = 1/30$.

b) Sketch the approximated solution with respect to x obtained in part (a).

c) Discuss the stability of the solution based on the results obtained in part (b).