University of Ruhuna

Bachelor of Science General Degree
Level III (Semester I) Examination - November 2021

Subject: Mathematics
Course Unit: MAT311β (Group Theory)

Time :Two (02) Hours

Answer All Questions.

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1. a) Consider the set $G' = \{(a,b) | a, b \in \mathbb{R}, a \neq 0\}$ and define $*$ on G' by

$$(a,b) * (c,d) = (ac, ad + b)$$

(i) Show that G' forms a group under the operation $*$. [50 marks]

(ii) Does G' form an abelian group? Justify your answer. [10 marks]

b) (i) Let H be a non-empty subset of G.
Show that H is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$. [15 marks]
Show that the set $H = \{(1,x) | x \in \mathbb{R}\}$ is a subgroup of the group G'. [15 marks]

(ii) If H is a subgroup of G then show that the inverse of an element $a \in H$ is the same as the inverse of the same element regarded as an element of the group G. [10 marks]

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2. a) (i) Let $\alpha = (1\ 3\ 2)$ and $\beta = (4\ 2\ 1\ 3)$. Find $\alpha\beta^{-1}$ and $\alpha^2\beta\alpha^{-1}$. [30 marks]

(ii) Write the permutation $\theta = (1\ 4\ 2)(2\ 3\ 5)(1\ 3\ 4)$ as a product of disjoint cycles.
Is θ odd or even permutation? [20 marks]

b) (i) Let H be a subgroup of the group G. Show that G is equal to the union of all **right** cosets of H in G. [10 marks]

(ii) Consider the group $\mathbb{P}_3 = \{I_3, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ under composition of mapping. Let $H = \{I_3, (1\ 2)\}$ be a subgroup of $\mathbb{P}$.
Find all **right** cosets of H in $\mathbb{P}_3$. [30 marks]

(iii) Show that set $\mathbb{P}_3$ is equal to the union of all right cosets of H in $\mathbb{P}_3$. [10 marks]

Continued.

3. a) Show that a subgroup H of a group G is normal if and only if $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. [20 marks]

b) The map $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_{ab}(x) = ax - b,$$

where $a, b \in \mathbb{R}$ and $a \neq 0$. Let $G = \{f_{ab} | a, b \in \mathbb{R}, a \neq 0\}$ be a group under the composition of mappings.

(i) Find the identity element of G and the inverse of $f_{ab} \in G$. [30 marks]

(ii) Show that $H = \{f_{1b} | b \in \mathbb{R}\}$ is a normal subgroup of G. [30 marks]

c) Prove the following for any two subgroups H and K of a group G.

(i) $H \cap K$ is a subgroup of G. [10 marks]

(ii) If H is normal in G then $H \cap K$ is normal in K. [10 marks]

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4. a) Let G and G' be two groups and $f : G \to G'$ be a homomorphism. Define Ker f, the kernel of f. [10 marks]
Prove that f is one-one if and only if $Ker f = \{e\}$, where e is the identity of G. [30 marks]

b) Let $(\mathbb{R}^+, .)$ be a multiplicative group of positive real numbers and $(\mathbb{R}, +)$ be a additive group of real numbers.

(i) Show that the map $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = log_{10}(x)$$

is a homomorphism. [10 marks]

(ii) Find $Ker f$. [20 marks]
Hence show that f is one-one. [10 marks]

(iii) Is f an isomorphism? Justify your answer. [20 marks]
