University of Ruhuna
Bachelor of Science General Degree
Level III (Semester I) Examination – November 2021

Subject: Applied Mathematics

Course Unit: AMT313β (Mathematical Methods in Physics and Engineering)

Time: Two (02) Hours

Answer All Questions

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(1) (i) In the usual notation, prove that

(a) $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, and

(b) if $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

Obtain the Laplace transforms of the following functions:

$$g(t) = e^{4t}\sin 2t + e^{-3t}\sinh 3t\,, \qquad \text{and}$$

$$f(\mathrm{t}) = \begin{cases} 0 & \text{if } t < 2 \\ t^2 & \text{if } \mathrm{t} \geq 2 \end{cases}.$$

(50 Marks)

(ii) Define the Inverse Laplace transform f(t) of F(s), denoted by $\mathcal{L}^{-1}\{F(s)\}$.

Find

(a) $\mathcal{L}^{-1}\{\dfrac{5s}{s^2+4s+29}\}$,

(b) $\mathcal{L}^{-1}\{\dfrac{(s-2se^{-2s})}{s^2+9s}\}$,

(c) $\mathcal{L}^{-1}\{\dfrac{s}{(s^2+9)^2}\}$.

(50 Marks)

(2) (i) Show that ,

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

in the usual notation.

Solve the following Initial Value Problem

$$y'' - y' = \cos(3t);\ \mathrm{y(0)=-9 \text{ and } y'(0)=0}$$

by using the Laplace Transform.

(50 Marks)

(ii) Use the Laplace transform method to solve the following system of differential equations for $x_1(t)$ and $x_2(t)$

$$x_1' = 5x_1 - 2x_2 + 3$$
$$x_2' = 3x_1 + t$$

subject to the initial conditions

$$x_1(0) = -1 \text{ and } x_2(0) = 1.$$

(50 Marks)

(3) (i) The Fourier series of a periodic function *f(t)* with period 2T is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{T};$$

where a_0, a_n and b_n are

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(t)dt, \quad a_n = \frac{1}{T}\int_{-T}^{T} f(t)\cos\frac{n\pi t}{T}dt \quad \text{and} \quad b_n = \frac{1}{T}\int_{-T}^{T} f(t)\sin\frac{n\pi t}{T}dt.$$

Find the Fourier series expansion of the function defined by

$$f(t) = \begin{cases} 4t & 0 < t < 1 \\ 1 & 1 \le t < 2 \end{cases}$$
$$f(t+2) = f(t).$$

Show that

$$f(1) = \frac{5}{2}.$$

Hence, deduce that

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}.$$

(60 Marks)

(ii) The definition of the Gamma function is given by

$$\Gamma(\lambda+1) = \int_0^{\infty} e^{-t} t^{\lambda} dt \quad \text{for all } \lambda > 1.$$

Obtain the recursive formula for the Gamma function as

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda).$$

Hence, evaluate the integral

$$\int_0^{\infty} e^{-4x^2} x^5 dx.$$

(40 Marks)

(4) (i) Show that the separable solution of the one dimensional heat equation

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2},$$

can be written, in the usual notation, in the form of

$$w(x,t) = \sum_{n=0}^{\infty} B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

along with the boundary conditions $w(0,t) = 0$, and $w(L,t) = 0$ for all time t.

Assuming that w is separable in x and t, solve the following one dimensional heat equation:

$$\frac{\partial w}{\partial t} = 3\frac{\partial^2 w}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

under the conditions

$$w(0,t) = w(\pi, t) = 0, \quad t > 0,$$
$$w(x,0) = 2\sin x + 3\sin 3x - 4\sin 4x, \quad 0 < x < \pi.$$

(60 Marks)

(ii) The Beta function is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \text{ for any real numbers } p, q > 0.$$

Using the above definition and the relations:

$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

evaluate the following integrals:

(a) $\displaystyle\int_0^3 x^2\sqrt{3-x}\,dx.$ (b) $\displaystyle\int_0^\infty \frac{x^2(1-x^3)}{(1+x)^8}\,dx.$

(40 Marks)

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