University of Ruhuna
Bachelor of Science Special Degree
Level I (Semester I) Examination –November 2021

Subject: Mathematics

Course Unit: MSP3144 (Mathematical Methods in Physics and Engineering)

Time: Three (03) Hours

Answer All Questions

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(1) (i) Prove, in the usual notation, that

(a) $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ and

(b) If $g(t) = f(t-a)u(t-a)$ then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

(20) Marks

(ii) Find the Laplace transform of the following:

(a) $f(t) = e^{4t} \sin 2t + e^{-3t} \sinh 3t$

(b) $f(t) = \begin{cases} 0 & \text{if } t < 2 \\ t^2 & \text{if } t \geq 2 \end{cases}$

(20) Marks

(iii) Find the following inverse Laplace Transforms:

(a) $\mathcal{L}^{-1}\{\frac{5s}{s^2+4s+29}\}$

(b) $\mathcal{L}^{-1}\{\frac{(s-2se^{-2s})}{s^2+9s}\}$.

(c) $\mathcal{L}^{-1}\{\frac{s}{(s^2+9)^2}\}$.

(30) Marks

(iii) Using the Laplace transforms, find the solution of the initial value problem

$$y'' - y' = \cos(3t); \quad y(0) = 9, y'(0) = 0.$$

(30) Marks

(2) (i) Let $f(t)$ be defined on the interval $(-T,T)$ and out side this interval $f(t)$ satisfies $f(t+2T)=f(t)$. The Fourier series of $f(t)$ is given by

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{T}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{T};$$

where a_0, a_n and b_n are given by

$$a_0=\frac{1}{T}\int_{-T}^{T}f(t)dt,\quad a_n=\frac{1}{T}\int_{-T}^{T}f(t)\cos\frac{n\pi t}{T}dt\quad \text{and}\quad b_n=\frac{1}{T}\int_{-T}^{T}f(t)\sin\frac{n\pi t}{T}dt.$$

(a) Find the Fourier series of the function

$$f(t)=\begin{cases}4t & 0<t<1\\ 1 & 1\le t<2\end{cases}$$

with $\quad f(t+2)=f(t)$.

(30) Marks

(b) Hence, deduce that

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}.$$

(20) Marks

(ii) The definitions of Gamma and Beta functions are given by

$$\Gamma(n+1)=\int_0^{\infty}e^{-t}t^n dt \quad \text{for all } n>1 \quad \text{and}$$

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \quad \text{for any real numbers } p,q>0$$

respectively.

Using the above definitions, and the relations

$$B(p,q)=\int_0^{\infty}\frac{x^{p-1}}{(1+x)^{p+q}}dx=2\int_{\phi=0}^{\frac{\pi}{2}}\cos^{2p-1}\phi\sin^{2q-1}\phi d\phi=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

evaluate the following integrals:

(a) $\displaystyle\int_0^{\infty}e^{-4x^2}x^5dx.$ $\qquad$ (b) $\displaystyle\int_0^3 x^2\sqrt{3-x}dx.$

(c) $\displaystyle\int_{\phi=0}^{\frac{\pi}{2}}\cos^3\phi\sin^5\phi\, d\phi.$ $\qquad$ (d) $\displaystyle\int_0^{\infty}\frac{x^2(1-x^3)}{(1+x)^8}dx.$

(50) Marks

(3) (i)

(a) Show that the separable solution of the one-dimensional heat equation

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}$$

can be written, in the usual notation, in the form of

$$w(x,t) = \sum_{n=0}^{\infty} B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin(\frac{n\pi x}{L})$$

subject to the boundary conditions $w(0,t) = 0, w(L,t) = 0$ for all time t.

(25) Marks

(b) Assuming that w is separable in x and t, solve the following one-dimensional heat equation:

$$\frac{\partial w}{\partial t} = 3\frac{\partial^2 w}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

under the conditions

$$w(0,t) = w(\pi,t) = 0, \quad t > 0,$$
$$w(x,0) = 2\sin x + 3\sin 3x - 4\sin 4x, \quad 0 < x < \pi.$$

(25) Marks

(ii) Consider the inhomogeneous one-dimensional wave equation given by

$$\frac{\partial^2 w}{\partial t^2} = 4\frac{\partial^2 w}{\partial x^2} + 2\sin\left(\frac{\pi x}{L}\right), \quad 0 < x < L$$

subject to the boundary conditions $w(0,t) = 0, w(L,t) = 0$ for all time t and

initial conditions $w(x,0) = 0$ and $\frac{\partial w}{\partial t}(x,0) = 0.$

(a) Taking the Laplace transform of both side and using the initial conditions, obtain the following equation

$$\frac{\partial^2 W(x,s)}{\partial x^2} - \frac{s^2}{4}W(x,s) = -\frac{1}{2s}\sin\left(\frac{\pi x}{L}\right), \quad 0 < x < L \text{ where } W(x,s) = \mathcal{L}(w(x,t)).$$

(20) Marks

(b) Show that the solution for the equation in part (a) is in the form

$$W(x,s) = C_1 e^{\frac{s}{2}x} + C_2 e^{-\frac{s}{2}x} + \frac{2}{s}\frac{\sin\left(\frac{\pi x}{L}\right)}{\left(s^2 + \frac{4\pi^2}{L^2}\right)}, \quad 0 < x < L.$$

(10) Marks

(c) Using the boundary conditions show that $C_1 = C_2 = 0.$

(10) Marks

(d) Taking the inverse Laplace transform for the solution in part (b) obtain the solution to the one-dimensional wave equation

$$w(x,t) = \frac{L^2}{2\pi^2}\sin(\frac{\pi x}{L})\left(1-\cos\frac{2\pi}{L}t\right) \quad t>0.$$

(10) Marks

(4) (i) Show that $y = 3$ is a particular solution of the Riccati Differential equation

$$\frac{dy}{dx} = y^2 - y - 6.$$

Substituting $y = 3 + \frac{1}{z}$ solve the above differential equation.

(25) Marks

(ii) The generating function of the Legendre polynomials $P_n(x)$ is defined by

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Show that

$$\sum_{n=0}^{\infty} t^n P_n(0) = \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} t^{2m}.$$

Deduce that

$$P_{2m}(0) - P_{2(m+1)}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}\left[1 + \frac{2m+1}{2(m+1)}\right].$$

$$\left\{\text{You may assume that } (1+t)^{-\frac{1}{2}} = \left(\sum_{m=0}^{\infty}(-1)^m \frac{(2m-1)(2m-3)\ldots 3.1}{2^{2m}(m!)} t^{2m}\right)\right\}$$

(25) Marks

(iii) The Bessel function $J_n(x)$ satisfies the equation:

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x).$$

Using the above equation, show that

(a) $\int J_1(x)dx = -J_0(x) + Const.$

(b) $\int J_3(x)dx = -J_2(x) - 2x^{-1}J_1(x) + Const.$

(25) Marks

(iv) The Hermit polynomial $H_n(x)$ of degree n is defined by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!},$$ for all integral values of ν and all real values of x.

Prove, in the usual notation, that

(a) $H_n(x) = (-1)^k \dfrac{n!\ (2x)^{n-2k}}{k!\ (n-2k)!}$.

(b) $H_{2m}(0) = (-1)^m 2^{2m} \left(\dfrac{1}{2}\right)_m$.

(25) Marks

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