UNIVERSITY OF RUHUNA

Faculty of Engineering

Mid-Semester 5 Examination in Engineering: June 2014

Module Number: IS5310 Module Name: Complex Analysis and Mathematical Methods

[Two hours]

[Answer **all questions**, each question carries **5** marks]

Q1. a)

(i) Show that the limit

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

does not exist. [1.0 Mark]

(ii) Show that the function

$$f(z) = \begin{cases} \dfrac{z \operatorname{Re} z}{|z|} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

is not differentiable at the origin. [1.0 Mark]

b)

(i) Show that the function given by $f(z) = e^{x^2 + 2ixy - y^2}$ is analytic and find its derivative. [1.5 Marks]

(ii) Let

$$f(z) = \begin{cases} \dfrac{x^{4/3} y^{5/3} + i y^{4/3} x^{5/3}}{x^2 + y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

Show that $f'(0)$ does not exist and yet $f(z)$ satisfies Cauchy-Riemann equations at $z = 0$. [1.5 Marks]

Q2. a) Find the analytic function $f(z) = u(x, y) + iv(x, y)$ given that

$$u - v = e^x (\cos y - \sin y).$$ [2.0 Marks]

b) Given that $v(x, y) = x^4 - 6x^2 y^2 + y^4$, find $f(z)$ in the form $u(x, y) + iv(x, y)$, where $u(x, y)$ is to be determined, such that $f(z)$ is analytic. Show also that u and v satisfy Laplace's equation. [3.0 Marks]

Q3. Let $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ be a piecewise smooth curve (contour) C, and $z'(t) = x'(t) + iy'(t)$, where $'$ denotes differentiation with respect to t. Then show that

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t)\, dt.$$ [1.0 Mark]

Evaluate the integral

$$\int_C z^2\,dz,$$

when C is given by

a) $z(t) = \begin{cases} 2t & ;0 \le t \le 1 \\ 2 + i(t-1) & ;1 \le t \le 2 \end{cases}$

[2.0 Marks]

b) The straight-line segment joining (0,0) to the point (4,1) on the complex plane.

[2.0 Marks]

Q4. a) If $f(z) = u(x,y) + iv(x,y)$ is continuous on the contour C, then

$$\left|\int_C f(z)\,dz\right| \le ML,$$

where L is the length of the contour C given by $z(t) = x(t) + iy(t), a \le t \le b$ and $M > 0$ is an upper bound for the modulus $|f(z)|$on C; that is $|f(z)| \le M$ for all $z \in C$. Using the above inequality, show that

(i)

$$\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{\pi}{3},$$

where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

[1.5 Marks]

(ii)

$$\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{1}{2\sqrt{5}},$$

where C is the straight-line segment from $z = 2$ to $z = 2 + i$

[Hint: You may use a geometric argument]

[1.5 Marks]

b) Let $f(z) = (z - z_0)^n$, where n is an integer and z_0 is a constant.

Show that

$$\int_C f(z)dz = \begin{cases} 2\pi i & ;n = -1 \\ 0 & ;n \ne -1, \end{cases}$$

where C is the circle $|z - z_0| = r$.

[2.0 Marks]