UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: May 2023

Module Number: IS5306 **Module Name: Numerical Methods**

[Three hours]

[Answer **all questions,** each question carries 12 marks]

Q1. a) Obtain the Newton Rapson's formula using Taylor's series.

[2 Marks]

b) Use Newton rapson's formula to find the approximate root of the following equation for two iterations:

$$f(x) = \sin x - \frac{(x+1)}{(x-1)}$$

with initial point $x_0 = -0.2$

Also, find the iterative error at each step.

[4 Marks]

c) Assume that you are working for a start-up computer assembly company and have been asked to determine the minimum number of computers that the shop will have to sell to make a profit. The equation that gives the minimum number of computers to be sold after considering the total costs and the total sales is

$$f(n) = 30n^{1.5} - 825n + 35000 = 0$$

i.) Use the bisection method with three iterations to estimate the root of the above equation to find the minimum number of computers that need to be sold to make a profit. Assume that the starting interval is [50,100].

ii.) Find the absolute relative approximate error at the end of each iteration.

[6 Marks]

Q2. a) The Lagrangian interpolating polynomial of degree n that passes through $n+1$ data points (x_0,y_0), (x_1,y_1),............, (x_n,y_n) is defined as $P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$.

where, $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$.

Show that $L_i(x_j) = 1$ when $j = i$ and

$L_i(x_j) = 0$ when $j \neq i$.

[2 Marks]

b) A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $35cm \times 20\ cm$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1: The coordinates of the holes on the plate .

Points	1	2	3	4	5	6
$x(cm)$	5.00	6.25	6.75	7.81	10.20	12.60
$y(cm)$	9.2	8.1	6.0	5.0	4.5	6.0

i.) If the laser is traversing from $x = 5.00$ to $x = 6.25$ in a linear path, what is the value of y at $x = 6.00$ using the Lagrangian method and a first order polynomial?

[2 Marks]

ii.) If the laser is traversing from $x = 5.00$ to $x = 6.25$ to $x = 6.75$ in a quadratic path, what is the value of y at $x = 6.00$ using a second order Lagrange polynomial?

[3 Marks]

iii.) Find the absolute relative approximate error for the second order polynomial approximation.

[1 Mark]

c) Consider the given the simultaneous linear quations.

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using Gauss elimination method.

[4 Marks]

Q3. a) Derive Trapezoidal and Simpson's rule formula using a Newtons formula.

[3 Marks]

b) The vertical distance in meters covered by a rocket from $t = 5$ to $t = 35$ seconds is given by,

$$x = \int_{5}^{35} f(t)\, dt$$

where, $f(t) = 2000 \ln\left[\dfrac{140000}{140000 - 2100t}\right] - 9.8t$

i.) Use the composite Simpson's rule with six subintervals to find the approximate value of x.

ii.) If the function $f(t)$ shown is known to have a fourth derivative with the property that, $|f^4(t)| < 0.012$ for $5 \le t \le 35$, determine how many subintervals are required so that the composite Simpson's rule used to approximate x, incurs an error less than 0.02.

[5 Marks]

c) The first level of processing what we see involves detecting edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as,

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases} \quad \text{need to be found.}$$

i.) Calculate the functions 1st derivative $f'(x)$ at $x = 0.1$ for $a = 0.12$, by using the central difference approximation. Use a step size of $h = 0.05$.

ii.) Calculate the functions 2nd derivative $f''(x)$ at $x = 0.1$ for $a = 0.12$, by using the central difference approximation. Use a step size of $h = 0.05$.

[4 Marks]

Q4. a) Briefly explain the following by giving an example for each item.

i.) Ordinary differential equations (ODE)
ii.) Partial differential Equations (PDE)
iii.) Initial value Problem (IVP)
iv.) Boundary value problem (BVP)

[3 Marks]

b) Explain the method of solving initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

by using Euler's method.

[3 Marks]

c) A polluted lake has an initial concentration of a bacteria of 10^5 parts/m^3, while the acceptable level is only 10^3 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.05C = 0$$

Using the Runge-Kutta 2nd order method, find the concentration of the pollutant after 5 weeks. Take a step size of 2.5 weeks.

[6 Marks]

Q5. a) Classify the following equations as linear or non-linear, and state their order.

i.) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$$

ii.) $$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

iii.) $$\alpha \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

iv.) $2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial x} + \cos(2t) = 0$

[3 Marks]

b) Solve $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ for $0 < x \leq 5$ and $0 < t \leq 0.2$

with $u(0,t) = 0$; $u(5,t) = 50$ and

$$u(x,0) = \begin{cases} 20x & for \quad 0 < x \leq 3 \\ 50 & for \quad 3 < x \leq 5 \end{cases}.$$

Use, $h = 1.0$ and $k = 0.1$, where h and k are step sizes along x and t axes respectively.

[6 Marks]

c) Briefly explain the procedure of finite difference solution technique for solving Laplace equation $\left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0\right)$ highlighting any differences with the solution technique used in part b).

[3 Marks]