UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 1 Examination in Engineering: December 2023

Module Number: IS1402 (C-18) **Module Name: Mathematical Fundamentals for Engineers**

[Three hours]

[Answer **all questions,** each question carries twelve marks]

Q1 a) If both the product and summation of two complex numbers, denoted as z and w, are real, show that either both z and w are real numbers or they are complex conjugates of one another.

[1 Mark]

b) i.) If $z = x + iy$ is a non-zero complex number, where x and y are real numbers, using an Argand diagram, explain what is meant by the modulus, argument, and the principal argument of z.

ii.) Find the modulus and argument of

$$z = \frac{5 + \sqrt{3}i}{2 - \sqrt{3}i}$$

[3 Marks]

c) i.) State the De Moivre's theorem for fractional powers of the type $1/p$, where $p \in \mathbb{N}$.

ii.) Show that summation of all the n^{th} roots of unity is 0.

Hence, show that

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \dots\dots + \cos\left(\frac{2(n-1)\pi}{n}\right) = -1$$

[4 Marks]

d) i) Use De Moivre's theorem to express $\sin^5\theta$ in terms of sines of multiples of θ.

Hence, evaluate

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^2\theta d\theta$$

[4 Marks]

Q2 a) Briefly explain the following matrices by giving a non-trivial example for each.

i.) Upper triangular matrix

ii.) Skew-symmetric matrix

iii.) Orthogonal matrix

iv.) Idempotent matrix

[4 Marks]

b) Use elementary transformation method to evaluate the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix}.$$

Hence, solve the system

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 1 \\ -3 & 6 \end{bmatrix}.$$

[4 Marks]

c) i.) Define an Augmented matrix.

ii.) What is it meant by a system of linear equations is consistent?

iii.) Find the values of α and β such that the system given below system is inconsistent.

$$\begin{aligned} x + 2y - z &= 3 \\ x - 2y + 2z &= \alpha \\ 2x + \beta y + z &= 2 \end{aligned}$$

[4 Marks]

Q3 a) i.) Explain what is meant by a function 'f is one to one'.

ii.) Show that, if f and g are one to one, so is $f \circ g$.

[2 Marks]

b) State whether the following functions are continuous at $x = 0$. Explain your answer.

i.) $f(x) = [x]$

ii.)
$$f(x) = \begin{cases} (1 + 2x)^{\frac{1}{x}} & ; \ x \neq 0 \\ e^2 & ; \ x = 0 \end{cases}$$
(Hint: $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$)

iii.)
$$f(x) = \begin{cases} 1; & x \neq 0 \\ -1; & x = 0 \end{cases}$$

iv.)
$$f(x) = \frac{2e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 2}$$

[6 Marks]

c) Evaluate the following limits

i.) $\lim_{x \to 0} \dfrac{\cos 2x - 1}{x \tan x}$

ii.) $\lim_{x \to \frac{\pi}{2}} \dfrac{\cos 3x - \cos 5x}{\cos 3x + \cos 5x}$

[2 Marks]

d) Sketch the graph of $y = 2|x - 1| - [x - 2] + |2x + 1|$

[2 Marks]

Q4 a) i.) State the Rolle's theorem

ii.) State the Mean Value theorem

iii.) If f and g are continuous on $[a, b]$, differentiable, and $g'(x) \neq 0$ on (a, b), show that there exist $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

iv) Hence, show that for $\frac{1}{2} < a < b < 2$, there exist $c \in (a, b)$ such that

$$\frac{b - a}{2} < \ln(b) - \ln(a) < 2(b - a)$$

Deduce that,

$$\frac{1}{4} < \ln(1.5) < 1$$

[6 Marks]

b) i) State and prove the Euler's theorem on homogeneous functions

ii) Let

$$u = \ln\left(\frac{x^4 - y^4}{x + y}\right)$$

Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

[4 Marks]

c) Write down the power series expansion of $f(x) = \sin x - \ln(x + 1)$ about $x = 0$.

[2 Marks]

Q5 a) i) Briefly explain what is meant by 'position vector' and 'unit vector'.

ii.) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, write down the modulus of $\mathbf{r}$ in terms of x, y and z.

iii) Find the modulus of the vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

[3 Marks]

b) Let the position vectors of three points A, B and C are,
$\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ respectively.

i.) Determine whether the points A, B and C are collinear or not.

ii.) Find the unit normal vector to $\mathbf{a}$ and $\mathbf{c}$.

iii.) Compute $\mathbf{b} \cdot \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

[4 Marks]

c) i) If the position vector of any arbitrary point (x, y, z) is denoted by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{r}| = r$, show that

$$\nabla\left(\frac{\mathbf{r}}{r^2}\right) = \frac{1}{r^2}$$

ii.) Find the directional derivative of $\emptyset = x^2y - xz^2$ in the direction of $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ at the point $(1, -1, -1)$.

iii.) Evaluate $\nabla \cdot (\nabla \times \mathbf{A}))$, where $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$.

[5 Marks]