University of Ruhuna

B.Sc. General Degree Level I (Semester I) Examination – July 2015

Subject: Mathematics

Course Unit: MAT 111β/MAM1113 (Vector Analysis)

Time : Tw0 (02) Hours

Answer 04 (Four) questions only

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(01) (a) Prove that the lines

$$\frac{x-5}{-3}=\frac{y+2}{1}=\frac{z-3}{-1} \text{ and}$$

$$\frac{x-10}{4}=\frac{y+3}{-1}=\frac{z-6}{2} \text{ intersect.}$$

Find their point of intersection and the equation of the plane through them.

(b) Find the shortest distance between the two skew lines

$\underline{r}=\underline{a}+\lambda\underline{b}$ and $\underline{r}=\underline{c}+\mu\underline{d}$; where λ and μ are parameters.

Hence find the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{1} \text{ and}$$

$$\frac{x-5}{3}=\frac{y-5}{2}=\frac{z+1}{-5}.$$

(02) (a) Write down an expression for *grad* ϕ, the gradient of a scalar field $\phi(x,y,z)$ in cartesian co-ordinates.
Show that *grad* ϕ is normal to the surface $\phi(x,y,z)=constant$ at $p(x,y,z)$ on the surface.

(b) Find the equations of the tangent plane and the normal line to the surface $xz^2+x^2y=z+1$ at the point (1,-3,2).

(c) Find the values of the constants a,b,c so that the directional derivatives of $\phi=ax^2+by^2+cz^2$ at (1,1,2) has a maximum magnitude 4 in the direction parallel to y axis.

(d) Show that $\underline{F} = (2x + yz)\underline{i} + (4y + zx)\underline{j} - (6z - xy)\underline{k}$ is solenoidal as well as irrotational.

Find a scalar potential ϕ such that $\underline{F} = \underline{\nabla}\phi$.

(03) (a) For a scalar field $\phi(x,y,z)$ and a vector field $\underline{A}(x,y,z)$, obtain the following results:

(i) $\underline{\nabla}.(\phi\underline{A}) = \phi\underline{\nabla}.\underline{A} + \underline{\nabla}\phi..\underline{A};$

(ii) $\underline{\nabla} \wedge (\phi\underline{A}) = \phi\underline{\nabla} \wedge \underline{A} + \underline{\nabla}\phi.\underline{\wedge}\,\underline{A}.;$

where $\underline{\nabla} == \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}.$

(b) If $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $|\underline{r}| = r$ and $\underline{a}, \underline{b}$ are constant vectors, show that

(i) $\underline{\nabla}.(r^n\underline{r}) = (n+3)r^n$

(ii) $\underline{\nabla}.(\underline{\nabla}r^n) = n(n+1)r^{n-2}$

(iii) $\underline{\nabla} \wedge (\underline{\nabla}r^n) = \underline{0}$

(iv) $\underline{\nabla}[\underline{r}, \underline{a}, \underline{b}] = \underline{a} \wedge \underline{b}$

(v) $\underline{\nabla} \wedge \left(\frac{\underline{a} \wedge \underline{r}}{r^3}\right) = -\frac{\underline{a}}{r^3} + \frac{3\underline{r}(\underline{a}.\underline{r})}{r^5}.$

(04) (a) State the Stoke's theorem. Hence deduce in the usual notation that,

$$\oint_c (\phi dx + \psi dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy;$$

the Green's theorem in the plane.

(b) Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_c (xdy - yxdy).$$

Hence find the area enclosed by the curves $y=x$ and $y^2=4x$.

(c) Verify Green's theorem in the plane for

$\oint_c [(x^2 + y^2)dx - 2xydy]$; where C is the rectangle in the xy plane bounded by $y=0$, $y=b$, $x=0$ and $x=a$.

(05) (a) State the divergence theorem.

Apply the divergence theorem to show that

(i) $\int_S (\nabla r^2).d\underline{S} = 6V$

(ii) $\int_S (\frac{\underline{r}}{r^2}).d\underline{S} = \int_V \frac{1}{r^2} dV$

(iii) $\int_S (r^m \underline{r}).d\underline{S} = (m+3)\int_V r^m dV.$

(b) Verify divergence theorem for $\underline{F} = 4x\underline{i} - 2y^2\underline{j} + z^2\underline{k}$ taken over the closed region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

(06) Let ψ be a scalar field and $\underline{A}$ be a vector field. In the usual notation, obtain the expressions for $\underline{\nabla\psi}$ and div $\underline{A}$ in orthogonal curvilinear co-ordinates (u_1, u_2, u_3).

Show that the system of co-ordinates (u, v, ϕ) defined by $x = uv\cos\phi$, $y = uv\sin\phi$, $z = \frac{1}{2}(u^2 - v^2)$ form an orthogonal system.

Find $\underline{\nabla\psi}$ and div $\underline{A}$ interms of the co-ordinates (u, v, ϕ).