University of Ruhuna

Bachelor of Science General Degree - Level II (Semester I) Examination - June/July 2015

Subject: Applied Mathematics
Course Unit: AMT 212β/MAM2123
Computational Mathematics **Time:: Two (02) Hours**

Answer four questions only.

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1. Consider the non-linear equation $f(x) = 0$.

   a) (i) State the necessary and sufficient conditions for f to apply the bisection method to find a root of the equation.

   (ii) Can the bisection method be used to find the roots of the equation $\sin x + 1 = 0$? Justify your answer.

   b) State clearly in stepwise form of the bisection method algorithm for solving the nonlinear equation $f(x) = 0$ in the interval $[a, b]$.

   c) (i) Derive the formula for the error bound in the n^{th} iteration step of the bisection method.

   (ii) How many iterations are needed to solve the equation $x^3 - x - 3 = 0$ on $[1, 2]$ using bisection method with guaranteed accuracy 10^{-5} ?

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2. a) Define the order of convergence (rate of convergence) α of the sequence x_k for $k = 0, 1, \ldots$ derived to obtain an approximation for the root x^* of a non linear equation.

   b) (i) State the Newton-Rapson formula (Newton's method) for finding an approximate root for the non linear equation $f(x) = 0$.

   (ii) Show that the Newton-Rapson method for the solution of $x^k e^x = 0$ is given by

   $$x_{n+1} = \frac{(k-1)x_n - x_n^2}{k + x_n} \quad n = 0, 1, 2, \ldots.$$

   Starting with $x_0 = 1$ calculate x_2 when $k = 2$.

   c) The cubic equation $x^3 - 2 = 0$ can be written in the form $x = g(x)$ in two ways, namely,

   $$x_{n+1} = g_1(x_n) = x_n^3 + x_n - 2$$
   $$x_{n+1} = g_2(x_n) = \frac{2 + 5x_n - x_n^3}{5}, \quad n \geq 0.$$

   Which iteration scheme convergres to the solution $2^{\frac{1}{3}}$ and why ? Use the convergent sequence to compute x_1 using $x_0 = 1.2$.

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3. a) For a function f the forward divided differences are given in the following table in the usual notation.

$x_0 = 0.0$	$f[x_0] = ?$		
		$f[x_0, x_1] = ?$	
$x_1 = 0.4$	$f[x_1] = ?$		$f[x_0, x_1, x_2] = \frac{50}{7}$
		$f[x_1, x_2] = 10$	
$x_2 = 0.7$	$f[x_2] = 6$		

(i) Determine the missing entries of the table.

(ii) Using the Newton's divided difference formula, find the interpolating polynomial $P_2(x)$ for the above data.

b) Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$ with $x_0 = a$, $x_n = b$, and $f \in C^{(n+1)}[a, b]$. Let $P(x)$ be the Lagrange polynomial interpolating the points such that $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

(i) Give the expression of $P(x)$ and its error formula.

(ii) Consider the case $n = 3$ and the following data:

x	0	1	2	3
$f(x)$	0	2	2	0

Obtain the Lagrange interpolating polynomial $P(x)$. Simplify your answer.

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4. a) A free cubic spline interpolating polynomial S for a function f, is defined on $[1, 3]$ by

$$S(x) = \begin{cases} S_0(x) = 2(x-1) - (x-1)^3 & 1 \le x < 2 \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & 2 \le x \le 3, \end{cases}$$

where a, b, c, $d \in \mathbb{R}$. Find a, b, c and d.

b) (i) Linear model for the data set (x_i, y_i) for $i = 1, 2, \ldots, n$ is given by $y_i = a_0 + a_1 x_i + e_i$, with error e_i for $i = 1, 2, \ldots, n$. Obtain the following estimates for a_0 and a_1 using the least squared approximation method.

$$\hat{a}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}, \qquad \hat{a}_0 = \bar{y} - \hat{a}_1 \bar{x},$$

where $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ and $\bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$.

(ii) Consider the following data set:

x_i	-3	-2	0	1
y_i	9	6	2	1

Using the least squared method, show that the best linear approximation to this data is $y = -2x + 2.5$.

Continued.

5. a) (i) Using the Taylor series expansion of a continuously differentiable function f at the point $x+h$ for some small $h > 0$, obtain the forward difference formula (FDF) of order h for the approximation of $f'(x)$. Describe the error term of the approximation.

(ii) Verify that the above approximation is exact for the derivatives of any polynomial up to degree 1.

b) Consider the 3-point formula of $f'(x_0)$ given in the usual notation by

$$f'(x_0) \approx \frac{1}{2h}(-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h))$$

(i) Explain briefly how you would obtain this 3-point formula.(Derivation is not required).

(ii) Using the formula with $h = 0.5$, approximate $f'(2)$, where $f(x) = x^2$.

(iii) Is your approximation exact ? Explain.

c) The derivative of a function can be expressed as

$$f'(x) = \frac{1}{2h}(f(x+h) - f(x-h)) - \frac{h^2}{6}f^{(3)}(x) - \frac{h^4}{120}f^{(5)}(x) - \ldots$$

with $h > 0$ small. Using this representation twice with h and $h/2$ and applying the Richardson extrapolation technique, derive a formula of order $O(h^4)$ to approximate $f'(x)$. What is the error in the term of h^4?

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6. a) (i) Obtain the Trapezoidal rule for the approximation of the integral $\int_a^b f(x)\,dx$ with the error term.

(ii) Use the Trapezoidal rule to approximate the integral $\int_0^2 \frac{1}{x+4}\,dx$.

b) The composite Trapezoidal rule is given by

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{j=1}^{n-1} f(a+jh)\right) - \left(\frac{b-a}{12}\right)h^2 f''(\zeta)$$

with $h = \frac{b-a}{n}$ and $n \geq 1$, where $\zeta \in (a, b)$.

Determine the values of n and h required to approximate the integral given in part (a)(ii) above within 10^{-3} and compute the integral using composite trapezoidal rule with $n = 4$.

c) (i) State the Simpson rule in approximating the integral $\int_a^b f(x)\,dx$.

(ii) Use the Simpson's rule to approximate the integral given in part (a)(ii).
