University of Ruhuna - Faculty of Science

Bachelor of Science General Degree - Level I
(Semester I) Examination - July 2016

Subject: Mathematics
Course Unit: MAT111β / MAM1113 (Vector Analysis)
Time: Two (02) Hours

Answer Four (04) questions only

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1. Let P be any point on a plane with position vector $\underline{r}$ relative to the origin O. Consider a point A on this plane with position vector $\underline{a}$ relative to the origin O. Suppose $\underline{n}$ is a vector normal to the plane.

Show that the vector equation of the plane can be written as $\underline{r}\cdot\underline{a}=\underline{a}\cdot\underline{n}$.

Now consider a point B with position vector $\underline{b}$ relative to the origin O such that BP is perpendicular to the plane. Show that the shortest distance, that is BP, to the plane $\underline{r}\cdot\underline{a}=\underline{a}\cdot\underline{n}$ from the point B is given by

$$\frac{|d-\underline{b}\cdot\underline{n}|}{|\underline{n}|},$$

where $\underline{r}\cdot\underline{a}=\underline{a}\cdot\underline{n}=d$.

Three points A, B and C have position vectors $\underline{i}+2\underline{j}-3\underline{k}, \underline{i}+5\underline{j}$ and $5\underline{i}+6\underline{j}-\underline{k}$ respectively, relative to an origin O.

(a) Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$ and find the area of the triangle ABC.

(b) Find the vector product $\overrightarrow{AB}\times\overrightarrow{BC}$. Hence find the equation of the plane ABC in the form $\underline{r}\cdot\underline{n}=p$, where p is a constant.

(c) The point D has position vector $4\underline{i}-\underline{j}+3\underline{k}$ relative to O. Find the shortest distance from the point D to the plane ABC. Hence show that the volume of the tetrahedron $ABCD$ is 21 (units).

(d) Obtain, in Cartesian form the equation of the plane Π which contains D and which has the property that for each point E in Π the volume of the tetrahedron $ABCE$ is still 21 (units).

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2. Define, in the usual notation,

(i) grad f of a scalar field $f(x,y,z)$

(ii) div $\underline{F}$ and curl $\underline{F}$ of a vector field $\underline{F}(x,y,z)=F_1\underline{i}+F_2\underline{j}+F_3\underline{k}$

(a) Consider the vector field $\underline{A}=xy^2\underline{i}+zy^2\underline{j}+xz^2\underline{k}$ on the sphere $x^2+y^2+z^2=14$.
Find

(i) the directional derivative of $\underline{A}^2$ at the point (2,0,3).

(ii) the outward unit normal vector to the sphere at the point (3,2,1)

(iii) the component of the directional derivative of part (i) in the direction given in part (ii).

(b) If a vector field $\underline{F}(x,y,z)$ is irrotational then there exists a scalar field $f(x,y,z)$ such that $\underline{F} = \text{grad} f$. Consider the vector field $\underline{F} = 2xye^z\underline{i} + x^2e^z\underline{j} + x^2ye^z\underline{k}$.

(i) Show that $\underline{F}$ is irrotational.

(ii) Show that the scalar function $f(x,y,z)$ is given by $f(x,y,z) = x^2ye^z + C$, where C is a constant.

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3. Verify the following results. Here $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $f(r)$ is a differentiable function. Furthermore, all notations have usual meanings.

(i) $\underline{\nabla}^2(\ln r) = \dfrac{1}{r^2}$

(ii) $\text{div}\left[\dfrac{f(r)}{r}\underline{r}\right] = \dfrac{1}{r^2}\dfrac{d}{dr}\left[r^2 f(r)\right]$

(iii) $\text{curl}\,[f(r)\underline{r}] = \underline{0}$

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4. (a) Explain what you mean by the line integral of a vector field $\underline{F}$ over a simple smooth curve C. A force $\underline{F} = 2x^2y\underline{i} + 3xy\underline{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along the curve given by $y = 4x^2$. Find the work done by the force.

(b) State the Green's theorem in the plane.

Verify Green's theorem for

$$\int_C (x-y)\,dx + 3xy\,dy,$$

where C is the boundary of the region enclosing $x^2 = 4y$ and $y^2 = 4x$.

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5. State the divergence theorem of Gauss.

If D is the region bounded by the closed cylinder $x^2 + y^2 = 16$, $z = 0$ and $z = 4$ and $\underline{F} = 3x^2\underline{i} + 6y^2\underline{j} + z\underline{k}$, verify the divergence theorem.

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6. State the Stokes' theorem.

Verify Stokes' theorem for the vector field $\underline{F} = (x^2 + y - 4)\underline{i} + 3xy\underline{j} + (2xz + z^2)\underline{k}$ over the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane and C is its boundary.
