University of Ruhuna

Bachelor of Science Special Degree Level I (Semester I) Examination - July 2016

Subject: Mathematics
Course Unit: MSP312β . (Real Analysis III)

Time :Two (02) Hours

Answer 04 Questions only.

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1. (a) Let $\mathbf{a} \in \mathbb{R}^n$, $r > 0$, and $\mathbf{A}$ be a subset of $\mathbb{R}^n$. Define the following terms:

(i) An open $n-$ball in $\mathbb{R}^n$ with the center $\mathbf{a}$ and the radius $r > 0$.

(ii) An interior point of $\mathbf{A}$.

(iii) An exterior point of $\mathbf{A}$.

(b) Prove that the Cauchy-Schwartz inequality

$$|\mathbf{x}.\mathbf{y}| \leq \| \mathbf{x} \| . \| \mathbf{y} \|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

(c) Using the Cauchy-Schwartz inequality prove that the triangle inequality

$$\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

(d) Show that

$$(a + b + c)^2 \leq 3\,(a^2 + b^2 + c^2)$$

where a, b, c are real numbers.

(e) Determine whether each of the following set is open or not.

(i) $\mathbf{A} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2\}$

(ii) $\mathbf{B} = \{\mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} \| \geq 2\}$.

Continued.

2. (a) Let $f : \mathbf{A} \to \mathbb{R}$ be a function defined on an open set $\mathbf{A} \subset \mathbb{R}^n$. Define the followings:

(i) $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$, and

(ii) f is continuous at a point $\mathbf{a}$,

where $\mathbf{a} \in \mathbf{A}$.

(b) Find the following limits:

(i) $$\lim_{(x,y)\to(0,0)} \frac{xy\,(x^2 - y^2)}{x^2 + y^2},$$

(ii) $$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2\,y^2 + 1} - 1}{x^2 + y^2}.$$

(c) Prove that the following limits do not exist.

(i) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2},$$

(ii) $$\lim_{(x,y)\to(0,1)} \frac{x}{x + y - 1}.$$

(d) Use part $(a)(ii)$, to show that

$$f(x,y) = \begin{cases} x\,\sin(\frac{1}{y}) + y\,\sin(\frac{1}{x}) & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

is continuous at $(0,0)$.

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3. Let $f(x,y)$ be a two vaiable function defined on a subset $\mathbf{A}$ of $\mathbb{R}^2$.

(a) Define the first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ of f.

(b) If

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

show that both partial derivetives exist at $(0,0)$, but the function is not continuous at $(0,0)$.

(c) Let $(a,b) \in \mathbf{A}$. If

(i) $\frac{\partial f}{\partial y}$ is continuous at (a,b),

(ii) $\frac{\partial f}{\partial x}$ exists at (a,b)

then show that f is differnctiable at (a,b).

(d) Show that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

is continuous, possesses partial derivatives but is not differentiable at the origin.

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4. (a) Let $z = x^3 - xy + y^3$, $x = r\cos(\theta)$, $y = r\sin(\theta)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(b) Show that $z = f(x,y)$ satisfies

$$x\,\frac{\partial z}{\partial x} = 2\,y\,\frac{\partial z}{\partial y},$$

where f is differentaible function.

(c) If $u_1, u_2, \ldots, u_n$ be n differntiable functions of n variables $x_1, x_2, \ldots, x_n$, then write down the Jacobian matrix.

(d) The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0,$$

are u, v, w. Prove that the Jacobian is given by,

Continued.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\ \frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$$

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5. (a) Explain briefly, how you find the extreme values of a two-variable function $f(x,y)$ defined on a subset in $\mathbb{R}^2$.

(b) Write down a necessary condition for $f(x,y)$ to have an extreme value at (a,b).

(c) Give an example for a function $f(x,y)$ that has an extreme value at $(0,0)$ but the partial derivatives f_x and f_y do not exist at $(0,0)$.

(d) Find the maxima and minima of the function

$$f(x,y) = x^3 + y^3 - 3\,x - 12\,y + 20.$$

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6. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differential function at (a,b,c). Show that

(i) the tangent plane to the surface $f(x,y,z) = 0$ at the point (a,b,c) is given by

$$f_x(a,b,c)\,(x-a) + f_y(a,b,c)\,(y-b) + f_z(a,b,c)\,(z-c) = 0, \text{ and}$$

(ii) the normal line to the surface at (a,b,c) is given by

$$\frac{(x-a)}{f_x(a,b,c)} = \frac{(y-b)}{f_y(a,b,c)} = \frac{(z-c)}{f_z(a,b,c)}.$$

(b) For the surface given by the equation $z = x^2 - 3\,y^2 + x\,y$, find the equations of the

(i) tangent plane

(ii) normal line

at the point $(1,1,-1)$ on the surface.

(c) Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellispe

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

is $\frac{8\,abc}{3\sqrt{3}}$.
