University of Ruhuna

Bachelor of Science General Degree

(Level III) Semester I Examination - July 2016

Subject :Applied Mathematics
Course unit: AMT311β /MAM3113 (Numerical Analysis)

Time :Two (02) Hours

Answer four (04) Questions only
Allowed to use calculators only supplied by the University.

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1. a) Define the norm $\| A \|_\infty$ of a matrix A of order n, in the usual notation.
For the matrix,

$$A = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 7 & -4 \\ 3 & 2 & 1 \end{pmatrix}, \text{ find } \| A \|_\infty.$$

b) Define the condition number, $\kappa(A)$ of a non singular matrix A.

$$\text{If } A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}, \text{ find } \kappa(A).$$

$$\text{You may use that, } A^{-1} = \begin{pmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{pmatrix}.$$

c) Solve the following system of linear equations using Gauss elimination method.

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4.$$

d) Apply Doolittle method and solve the following system of linear equations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.$$

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2. a) The system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, has an equivalent representation of the form $x = Tx + c$, where $T \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$.

(i) Show that $x = Tx + c$ has a unique solution, if $\| T \| < 1$ for some norm.

(ii) Let $x^* \in \mathbb{R}^n$ be the unique solution of the system $x = Tx + c$.
Consider the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the recurrence formula
$x^{(k+1)} = Tx^{(k)} + c, \quad k = 0, 1, 2, \ldots$ with the initial approximation $x^{(0)}$.
Show that $x^{(n)} - x^* = T^n(x^{(0)} - x^*)$.

b) (i) Consider the system of linear equations $Ax = b$ and the decomposition $A = L + D + U$, where L, D and U represent the lower triangular, diagonal and upper triangular parts of A respectively.
Use this decomposition to formulate the Gauss Seidel iteration method for solving $Ax = b$ and write down the general formula for the method.

(ii) Consider the system of linear equations

$$\begin{pmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Taking the initial approximation as $x^{(0)} = (1\ 1\ 1)^T$, find the second iterate $x^{(2)}$ using the Gauss Seidel iteration method.

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3. a) Consider the initial value problem $y' = f(t, y) = 1 + ty^2, \quad y(0) = 1$ on $D = \{(t, y) |\ 0 \le t \le \frac{3}{2},\ 0 \le y \le 2\}$.

(i) Show that f satisfies Lipschitz condition on D.

(ii) Find a Lipschitz constant.

b) Consider the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$.
Obtain the corresponding Picard iteration formula,
$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s))ds$; $n \ge 0$ for solving this problem.
Using this formula find $y_3(x)$ of the initial value problem $y' = 3(y + 1), \quad y(0) = 0$.

c) Using Taylor expansion derive the second order approximation scheme to find $y(x_{n+1})$, for the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$ with step size h.
Consider the initial value problem $y' = x + e^y, \quad y(0) = 1$.
Use the above derived formula to find the approximate value of $y(0.2)$ by taking $h = 0.1$.

Continued.

4. a) In the usual notation, write down the general form for the fourth order Runge Kutta method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ with step size h.

The fourth order classical Runge-Kutta method is described by the following Butcher's table.

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

Write down the corresponding Runge-Kutta scheme.
Apply this scheme to the initial value problem $y' = x + y$, $y(0) = 1$, with the step size 0.1 and obtain the approximate value of $y(0.2)$.

b) Consider the system of differential equations

$$x'(t) = xy + t$$
$$y'(t) = 2y - x^2$$

with initial conditions $x(1) = 0$ and $y(1) = 3$.
Find the approximate values of $x(1.2)$ and $y(1.2)$ using Explicit Euler's method with step size 0.1

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5. Let $u(x, t)$ be the solution of the heat equation, $u_{xx} = cu_t$ for $t > 0$ and $0 < x < a$ with boundary conditions $u(0, t) = T_0$ and $u(a, t) = T_1$, for $t > 0$ and, initial condition $u(x, 0) = f(x)$, for $0 \leq x \leq a$.

Consider the discretization $x_i = ih$, $i = 0, 1, 2, ..., n$ and $t_j = jk$, $j = 0, 1, 2, ...$, where $h = \frac{a}{n}$, $k > 0$ are sufficiently small step sizes in x and t directions respectively.
In the usual notation, derive, the finite difference scheme,

$$u_{ij+1} = ru_{i+1j} + (1 - 2r)u_{ij} + ru_{i-1j}, \text{ where } r = \frac{k}{ch^2}.$$

Using the above finite difference scheme with $h = 0.2$ and $k = 0.02$, solve the heat equation $u_{xx} = u_t$ for $0 < x < 1$ and $0 < t < 0.06$,
with boundary conditions $u(0, t) = u(1, t) = 0$, for $t > 0$ and
initial condition $u(x, 0) = sin2\pi x$, for $0 \leq x \leq 1$.

Continued.

6. Consider the partial differential equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y).$$

where A, B and C are functions of the variables x and y.
Explain how would you classify above equation as parabolic, elliptic and hyperbolic.

Consider the wave equation, $u_{tt} = c^2 u_{xx}$ for $0 < x < a$ and $t > 0$.
Show that it is of hyperbolic type.

Let $u(x,t)$ be the solution of the wave equation, $u_{tt} = c^2 u_{xx}$ for $0 < x < a$ and $t > 0$ subject to the initial conditions $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$ for $0 \le x \le a$ and boundary conditions $u(0,t) = \phi(t)$ and $u(a,t) = \phi(t)$ for $t \ge 0$.

Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ..$ where $h = \frac{a}{n}$; $k > 0$ are sufficiently small step sizes in x and t directions respectively.
In the usual notation, derive the explicit finite difference scheme,

$$u_{ij+1} = -u_{ij-1} + \alpha^2(u_{i+1j} + u_{i-1j}) + 2(1 - \alpha^2)u_{ij}, \text{ where } \alpha = \tfrac{ck}{h}.$$

Consider the equation, $u_{tt} = 16u_{xx}$, $0 < x < 5$ and $0 < t < 0.75$ with the initial conditions $u(x,0) = -x^3 + 5x^2 + 2$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for $0 < x < 5$ and boundary conditions $u(0,t) = u(5,t) = 0$ for $t \ge 0$.
Solve the equation using the finite difference scheme derived above, with $h = 1$ and $k = 0.25$.
