University of Ruhuna

Bachelor of Science Special Degree in Mathematics
Level I (Semester I) - Examination. August 2017

Subject: Mathematics
Course Unit: MSP3144
Mathematical Methods for Physics and Engineering Time:Three (03) Hours

Answer four (04) questions selecting at least one from Section B

Section A

1. a) (i) Define the exponential order of a function $f(t)$.

(ii) Determine whether the function $f(t) = \dfrac{1}{t-5}$ is of exponential order or not.

b) (i) Define $\mathcal{L}(f(t))$, the Laplace transforms of a function $f(t)$, denoted by $F(s)$.

(ii) Using the definition of Laplace transform, find $\mathcal{L}(3e^{-2t})$ and deduce the region for s.

c) If $\mathcal{L}(f(t)) = F(s)$, show that

(i) $\mathcal{L}(e^{at}f(t)) = F(s-a)$,

(ii) if $g(t) = \begin{cases} f(t-a); & t > a, \\ 0; & t \leq a, \end{cases}$
then $\mathcal{L}(g(t)) = e^{-as}F(s)$,

(iii) $\mathcal{L}(tf(t)) = -\dfrac{d}{ds}(F(s))$.

d) Using the Laplace transform $\mathcal{L}(e^{iat})$ obtain the Laplace transforms $\mathcal{L}(\sin(at))$ and $\mathcal{L}(\cos(at))$.

e) Let $f(t) = \begin{cases} t^2+1; & 0 \leq t < 1, \\ e^{-3t}+1; & 1 \leq t < 2, \\ 1; & t > 1. \end{cases}$

(i) Write down the function $f(t)$ in terms of unit step functions.

(ii) Using the appropriate results in part (c) or otherwise find $\mathcal{L}(f(t))$.

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2. a) (i) Define the inverse Laplace transform $\mathcal{L}^{-1}(F(s))$ of $F(s)$ denoted by $f(t)$.

(ii) Find the inverse Laplace transform $\mathcal{L}^{-1}\left(\dfrac{1}{s(s^2+1)}\right)$.

(iii) Use the derivative property of Laplace transforms to find $\mathcal{L}^{-1}(s^{-4})$.

b) (i) State the convolution theorem of Laplace transforms.

(ii) Use this theorem to find the inverse Laplace transform $\mathcal{L}^{-1}\left(\dfrac{1}{s^2(s^2+1)}\right)$.

c) Let $\mathcal{L}(y(t)) = Y(s)$. Show, in the usual notation, that

(i) $\mathcal{L}(y'(t)) = sY(s) - y(0)$.

(ii) $\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$.

d) Use the Laplace transform method to solve the initial value problem

$$y''(t) - y(t) = 1, \qquad y(0) = 0, \quad y'(0) = 1.$$

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3. a) Let $f(x) = \cos x + 2\sin(5x)$.

(i) Show that the function f is periodic with period 2π.

(ii) Determine whether the function f is even, odd or neither.

b) Let the Fourier series of the periodic function $f(x) = x$ in the interval $(-\pi, \pi)$ be given by, in the usual notation,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

(i) Find the Fourier coefficients a_0, a_n and b_n and obtain the Fourier series $f(x)$.

(ii) To what value does the Fourier series converge at $x = \frac{\pi}{2}$?

(iii) To what value does the Fourier series converge at $x = \pi$?

(iv) Evaluate the sum $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ of the Fourier series above with a minimum number of calculations.

c) Consider the Fourier series of complex form of the function $f(x) = 1 + x, \quad -\pi < x < \pi$, given by

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{inx}.$$

Show that $c_0 = 1$ and find $c_n, \quad n \neq 0$.

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4. a) (i) Define the gamma function $\Gamma(x)$, $x > 0$.

(ii) Using the definition of gamma function, show that $\Gamma(x+1) = x\Gamma(x)$.

(iii) Find $\Gamma\left(\frac{5}{2}\right)$ and $\Gamma\left(-\frac{1}{2}\right)$, provided $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(iv) Show that $\int_0^{\infty} x^3 e^{-4x}\, dx = \frac{3}{128}$.

b) (i) Define the beta function $B(m, n)$ for any $m, n > 0$.

(ii) Using beta and gamma functions with the substitution $x - 1 = 2y$ show that $\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \pi$.

c) The Bessel function $J_\nu(x)$ of order ν is given by, in the usual notation,

$$J_\nu(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k}.$$

Show that $J_{-n}(x) = (-1)^n J_n(x)$ for $n \in \mathbb{N}$.

d) Consider the differential equation $xy'' + 3y' + xy = 0$.

(i) Using the substitution $u = xy$ convert this differential equation into the Bessel equation of order 1.

(ii) Write down the Bessel function for the Bessel equation obtained in part(i) above and hence find a solution $y(x)$ for the given differential equation.

Section B

5. a) (i) Define the Fourier transform $\mathcal{F}(f(t))$ and inverse Fourier transform $\mathcal{F}^{-1}(F(\omega))$.

(ii) Let a function $f(t)$ which has a Fourier transform $F(\omega)$. Prove, in the usual notation, that

i. $\mathcal{F}\left(f(t)e^{i\omega_0 t}\right) = F(\omega - \omega_0)$,

ii. $\mathcal{F}(f(t - t_0)) = F(\omega)e^{-i\omega t_0}$.

b) Find the Fourier transform of the following functions

(i) $f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0, \alpha > 0 \end{cases}$

(ii) $f(t) = u_2(t)e^{-4t}$, where $u_2(t)$ is a unit step function.

(iii) $f(t) = e^{-4(t+1)}$.

c) Find the inverse Fourier transform of the function $F(\omega) = \dfrac{e^{2i\omega}}{2 + 3i}$.

d) Consider the initial value problem

$$u_t - (\sin t)u_x = 0, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x, 0) = \Phi(x), \quad -\infty < x < \infty.$$

(i) Using the Fourier transforms, show that the problem can be converted to an ordinary differential equation with the corresponding initial condition.

(ii) Show that, in the usual notation, $\hat{u}(\omega, t) = \hat{\Phi}(\omega, t)e^{i\omega - i\omega \cos t}$, where $\hat{u}$ and $\hat{\Phi}$ are the Fourier transforms of u and Φ respectively.

(iii) Considering the inverse Fourier transform, show that the solution of the problem is given by $u(x, t) = \Phi(x + 1 - \cos t)$.

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6. a) A series solution of the differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

is given by the Legendre polynomial

$$P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r(2n - 2r)!}{2^n r!(n - r)!(n - 2r)!} x^{n-2r},$$

where $[n/2] = n/2$ if n is even and $[n/2] = (n - 1)/2$ if n is odd.

(i) Use above formula to obtain the first three Legendre polynomilas $P_0(x)$, $P_1(x)$ and $P_2(x)$.

(ii) Express the polynomial $4x^2 - 3x + 2$ in terms of Legendre polynomials.

b) The generating function of the Legendre polynomials $P_n(x)$ is defined by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(t)t^n.$$

(i) By differentiatig the above formulae show that

$$\sum_{n=0}^{\infty}(x-t)P_n(x)t^n = \sum_{n=1}^{\infty}(1-2xt+t^2)nP_n(x)t^{n-1}.$$

(ii) Considering the coefficients of the powers of t^n show that

$$(2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nP_{n-1}(x).$$

c) Let $P_n(x)$ be the Legendre polynomial defined through the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}((x^2-1)^n)$$

and consider the result $P_n(x) = k_n x^n + Poly_{n-2}(x)$, where $Poly_{n-2}(x)$ is the polynomial of degree $n-2$ and k_n given by

$$k_n = \frac{(2n)!}{2^n(n!)^2} \text{ or equivalently by } k_n = \frac{2^n\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})}.$$

(i) Using one of the above formulae of k_n find the value α of the recursion relation

$$(n+1)P_{n+1}(x) = \alpha x P_n(x) - nP_{n-1}(x).$$

(ii) Using the Rodrigue's formula show that

$$P_k{}'(x) = \frac{1}{2^{k-1}(k-1)!}\frac{d^{k-1}}{dx^{k-1}}(((2k-1)x^2-1)(x^2-1)^{k-2})$$

and hence find the formula $P'_{n+1}(x)$.

(iii) Directly obtaining $P'_{n-1}(x)$ from the derivative of Rodrigue's formula and using the result in part (ii) above show that

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$
