University of Ruhuna
Bachelor of Science Special Degree
(Level I) Semester I Examination - August/September 2017

Subject : Mathematics
Course unit: MSP3254 (Numerical Methods with Applications)

Time :Three (03) Hours

Answer four (04) Questions only
Only the calculators provided by the University are allowed to use.

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1. a) Define, in the usual notation, the matrix norms, $\| A \|_1$, $\| A \|_2$, $\| A \|_\infty$ and the condition number, $\kappa(A)$ of a non singular matrix A of order n.

   Find $\| A \|_1$, $\| A \|_\infty$ and $\kappa(A)$ for the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{pmatrix}.$$

   b) Consider the system of linear equations $Ax = b$, where A is a non singular matrix. Suppose that $\bar{x}$ is an approximate solution to this system which satisfies $A\bar{x} = \bar{b}$. Show that $\frac{1}{\| A \| \| A^{-1} \|} \frac{\| r \|}{\| b \|} \leqslant \frac{\| x - \bar{x} \|}{\| x \|} \leqslant \| A \| \| A^{-1} \| \frac{\| r \|}{\| b \|}$; where $r = b - \bar{b}$

   c) Solve the following system of linear equations using Gauss elimination method.
   $2x + 4y - 6z = -8$
   $x + 3y + z = 10$
   $2x - 4y - 2z = -12$

   d) Apply Doolittle method to solve the following system of linear equations.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}.$$

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2. a) If A is a square matrix of order n such that $\| A \| < 1$, where $\| . \|$ is any matrix norm, then show that

(i) $(I - A)$ is invertible

(ii) $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$; where I is the identity matrix.

b) The system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, has an equivalent representation of the form $x = Tx + c$, where $T \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$.
Suppose that the system has a unique solution $x^* \in \mathbb{R}^n$.
Consider the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the recurrence relation $x^{(k+1)} = Tx^{(k)} + c$, where $k = 0, 1, 2, ...$ with the initial approximation $x^{(0)}$.
Show that $x^{(k)} - x^* = T^k(x^{(0)} - x^*)$.

c) (i) Consider the system of linear equations $Ax = b$ and the decomposition $A = L + D + U$, where L, D and U represent the lower triangular, diagonal and upper triangular parts of A respectively. Use this decomposition to obtain the general formula of jacobi iteration method for solving $Ax = b$.

(ii) Show that, if A is strictly diagonal dominant then the Jacobi method is convergent.

(iii) Consider the system of linear equations given by

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 8 \end{pmatrix}.$$

Taking the initial approximation as $x^{(0)} = (0\ 0\ 0)^T$, find the second iterate $x^{(2)}$ using the Jacobi iteration method.

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3. a) Consider the rectangular region $D = \{(t, y) | 0 \le t \le 5,\ -1 \le y \le 1\}$.
Let $f(t, y) = t^2 y - 1$ with $y(0) = 1$.
Does f satisfy the Lipschitz condition on D ? If so find a Lipschitz constant.

b) Consider the initial value problem $y' = xe^y$; $y(0) = 0$.
In the usual notation, calculate the Picard Iterations $y_1(x)$ and $y_2(x)$ for this initial value problem.

c) Consider the initial value problem $y' = f(x, y)$; $y(x_0) = y_0$ with one step θ method given by $y_{j+1} = y_j + h[\theta f(x_j, y_j) + (1 - \theta) f(x_{j+1}, y_{j+1})]$; $\theta \in [0, 1]$.
Considering both discrete and continuous evolutions, show that the order of consistence of the method is 2 if $\theta = \frac{1}{2}$.

d) Consider the initial value problem $y''(x) + 2y'(x) - 3y(x) = 6x$, with $y(0) = 0$ and $y'(0) = 1$.

(i) Transform this initial value problem into an equivalent system of first order differential equations.

(ii) Apply explicit Euler's method to the above system with step size 0.2, and find the approximate values for y' and y at $x = 0.2$ and $x = 0.4$.

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4. a) Describe the Predictor-Corrector technique in approximating the solution of an initial value problem using modified Euler's method and explicit Euler's method.
Apply the Predictor-Corrector method described above to the initial value problem $y' = 2y/x$; $y(1) = 2$ with step size $h = 0.25$, to obtain the approximate value of $y(1.25)$.

b) Using Taylor expansion derive the third order approximation scheme to find $y(x_{n+1})$, for the initial value problem $y' = f(x, y)$; $y(x_0) = y_0$ for the step size h.
Consider the initial value problem $y' = x^2 + y$; $y(0) = 1$.
Using the derived scheme above, find the approximate value of $y(0.1)$ taking the step size as 0.1.

c) In the usual notation, write down the general form of the fourth order Runge Kutta method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ with step size h.

The fourth order classical Runge-Kutta method is described by the following Butcher's table.

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	2/6	2/6	1/6

Write down the corresponding Runge-Kutta scheme.
Apply this scheme to the initial value problem $y' = x^2 + y^2$, $y(0) = 0$, with the step size 0.2 and obtain the approximate value of $y(0.4)$.

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5. Let $u(x, t)$ be the solution of the heat equation, $u_{xx} = cu_t$ for $t > 0$ and $0 < x < a$ with the boundary conditions $u(0, t) = T_0$ and $u(a, t) = T_1$, for $t > 0$ and, initial condition $u(x, 0) = f(x)$, for $0 \leq x \leq a$.

Consider the space and spatial discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ...$ where $h = \frac{a}{n}$, $k > 0$ are sufficiently small step sizes, respectively.

a) Derive, in the usual notation, the explicit finite difference scheme,
$u_{i,j+1} = ru_{i+1,j} + (1 - 2r)u_{i,j} + ru_{i-1,j}$; where $r = \frac{k}{ch^2}$.

Draw the stencil for the scheme.
Consider the heat equation $2u_{xx} = u_t$ for $0 < t < 1.5$ and $0 < x < 4$ with the boundary conditions $u(0, t) = u(4, t) = 0$, for $0 \leq t \leq 1.5$ and the initial condition $u(x, 0) = 50(4 - x)$, for $0 < x < 4$.

Using the above explicit finite difference scheme with $h = 1$ and $k = 0.25$, solve the heat equation $2u_{xx} = u_t$ for $0 < t < 1.5$ and $0 < x < 4$.

b) Derive, in the usual notation, Crank Nicholson implicit scheme,
$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = (2-2r)u_{i,j} + r(u_{i-1,j} + u_{i+1,j})$; where $r = \frac{k}{ch^2}$.
for solving the heat equation.

Consider the heat equation $u_{xx} = u_t$ for $t > 0$ and $0 < x < 4$
with the boundary conditions $u(0,t) = u(4,t) = 0$, for $t > 0$ and
the initial condition $u(x,0) = x(4-x)$, for $0 \leq x \leq 4$ with $h = k = 1$.
Obtain the matrix form of the discrete problem.

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6. a) Let $u(x,y)$ be the solution of the Laplace equation, $u_{xx} + u_{yy} = 0$ for $0 < x < a$ and $0 < y < b$ subject to the conditions that $u(x,y)$ are known at the boundary grid points.

Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $y_j = jk$, for $j = 0, 1, 2, .., m$ where $h = \frac{a}{n}, k = \frac{b}{m}$ are sufficiently small step sizes in x and y directions, respectively.

Assuming equal step sizes, derive the five-point finite difference formula,
$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$.

Consider the Laplace equation $u_{xx} + u_{yy} = 0$ over the square $x = 0, y = 0, x = 15$ and $y = 15$.
The boundary conditions are given by $u(x,0) = 0$, $u(0,y) = 0$, $u(x,15) = 100$ and $u(15,y) = 100$ with the step sizes $h = k = 3$. Construct the relevant equations to approximate unknown $u(x,y)$ at interior points.
(It is not necessary to solve equations.)

b) Let $u(x,t)$ be the solution of the wave equation, $u_{tt} = c^2 u_{xx}$ for $0 < x < a$ and $t > 0$ subject to the initial conditions $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$, for $0 \leq x \leq a$ and boundary conditions $u(0,t) = \phi(t)$ and $u(a,t) = \phi(t)$, for $t \geq 0$.

Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, .., m$ where $h = \frac{a}{n}, k = \frac{b}{m}$ are sufficiently small step sizes in x and t directions respectively.

Derive, in the usual notation, the explicit finite difference scheme,
$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i+1,j} + u_{i-1,j}) + 2(1-\alpha^2)u_{ij}$; where $\alpha = \frac{ck}{h}$.

Consider the wave equation, $u_{tt} = 4u_{xx}$ for $0 < x < 5$ and $0 < t < 2$ with the initial conditions $u(x,0) = x(5-x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, for $0 \leq x \leq 5$ and boundary conditions $u(0,t) = u(5,t) = 0$, for $0 \leq t \leq 2$.
Solve this wave equation using the explicit finite difference scheme derived above, with step sizes $h = 1$ and $k = 0.5$.
