University of Ruhuna

Bachelor of Science Special Degree
Level II (Semester I) Examination

August 2017

Subject: Mathematics
Course Unit: MSP4114 (Ring and Field Theory)
Time: Three (03) Hours

Answer Four (04) Questions only

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1. a) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$.
Prove that R is a ring under matrix addition and matrix multiplication.

b) Is it possible to have a ring without unity which has a subring with unity? Justify your answer using the ring R in part a).

c) Let $R' = \left\{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \right\}$ be a ring with respect to ordinary addition and multiplication. Prove that R' is an integral domain.
Does R' form a field? Justify your answer.

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2. a) Show that the ideal $A = \left\{ xf(x) + 2g(x) \mid f(x), g(x) \in \mathbb{Z}[x] \right\}$ is maximal in $\mathbb{Z}[x]$.

b) Let R be a ring. If M is an ideal of R such that $R \mid M$ is a field then show that M is maximal.

c) Let I and J be two ideals of the ring R. In the usual notation prove that

(i) $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$

(ii) $\sqrt{I} + \sqrt{J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}}$

d) Define a primary ideal.
Let I be an ideal of the ring R. If I is a primary ideal then show that every zero divisor of the quotient ring R is nilpotent.

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3. a) Let R be a commutative ring and suppose $px = 0$ for all $x \in R$, where p is a prime number. Show that the mapping $f : R \longrightarrow R$ defined by $f(x) = x^p,\ x \in R$ is a homomorphism.

b) Let I and J be two ideals of a ring R. Prove that
$(I + J) \mid I \cong J \mid I \cap J$.

c) Let R be a commutative ring with unity and $\phi : R \longrightarrow \dfrac{R}{I} \times \dfrac{R}{J}$ such that $\phi(x) = (x + I, x + J)$ be a homomorphism. If I and J are comaximal ideals then show that ϕ is onto.

Continued.

4. a) Let R be a commutative ring with unity. $a \sim b$ denotes that "a is associate of b", where $a, b \in R$. Show that "$\sim$" is an equivalance relation.

   b) Find the associates of $5 + 2i$ in $\mathbb{C}$, the ring of Gaussian integers.

   c) Define a Euclidean ring.
   Let a, b be two non-zero elements of a Euclidean ring R. In the usual notation, show that

   (i) if b is a unit of R, then $d(ab) = d(a)$.

   (ii) If b is not a unit of R, then $d(ab) > d(a)$.

   d) Show that in a Principal Ideal Domain, an element is prime if it is irreducible.

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5. a) Find the greatest common divisor of $x^5 + 2x^3 + x^2 + 2x$ and $x^4 + x^3 + x^2$ in the field of residue classes modulo 3.

   b) Prove that a polynomial $f(x)$ over a field F has a multiple zero in some extension K iff $f(x)$ and $f'(x)$ have a comman factor of positive degree in $F[x]$.

   c) Let R be an integral domain and $R[x]$, the set of polynomials over R be a ring. Show that $R[x]$ is an integral domain.

   d) Prove that if F is a field $F[x]$ may not be a field.

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6. a) Show that in a Boolean ring R, every prime ideal $P(\neq R)$ is maximal.

   b) Let I be an ideal of the ring R. If $R \mid I$ has no non-zero nilpotent elements then show that I is semiprime.

   c) Prove that in a ring a non-zero idempotent element cannot be nilpotent.

   d) Use contradiction method to show that $f(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$.
   Describe the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
