University of Ruhuna

Bachelor of Science General Degree Level II (Semester II) Examination - January 2018

Subject: Applied Mathematics/ Industrial Mathematics
Course Unit: AMT 221β/IMT 221β
Mathematical Modelling II

Time: Two (02) Hours

Answer ~~the~~ Four (04) questions only

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1. a) A swimming pool is initially filled with 400 m^3 fresh water. At time $t = 0$ water containing 50 g/m^3 of chlorine starts to flow into the swimming pool at a rate of 2 m^3 per minute. Well-mixed water is drained out from the pool at the same rate.

   (i) Set up an initial value problem for modelling this process.

   (ii) Find the time that would take the chlorine concentration within the pool reaches 25 g/m^3.

   b) Consider the initial value problem

$$y' = 2t(1+y), \quad y(0) = 0.$$

   (i) Compute Picard's iterates $y_n(t)$ for $n = 1, 2, 3$ with initial guess $y_0(t) = 0$.

   (ii) Considering the form of the sequence of iterates $\{y_n(t)\}_{n=0}^{\infty}$ obtain the exact solution $y(t)$.

   c) Use the Taylors series method of order 3 to obtain an approximate solution $y(h)$ to the initial value problem

$$y' = y\cos t, \quad y(0) = 1$$

   at time $t = h$, where $h > 0$ and small.

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2. a) (i) Describe the explicit Euler's method for solving the initial value problem,

$$y' = f(t, y), \quad y(t_0) = y_0.$$

   (ii) Use the Explicit Euler's method with step size $h = 0.5$ to solve the initial value problem,

$$y' = t^2 y, \quad y(0) = 1 \qquad \text{on } [0, 1].$$

   (iii) Apply the fully implicit Euler's formula, to approximate $y(0.5)$ of the initial value problem in above a)(ii).

b) The general formula for the second stage Runge-Kutta methods for solving the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ is given by, in the usual notation,

$$\begin{aligned} k_1 &= f(t_0, y_0) \\ k_2 &= f(t_0 + c_2 h, y_0 + a_{21} h k_1) \\ y_1 &= y_0 + h(b_1 k_1 + b_2 k_2), \end{aligned}$$

where c_2, a_{21}, b_2 and b_2 are constants to be determined such that

$$a_{21} c_2 = \frac{1}{2}, \qquad b_2 c_2 = \frac{1}{2}, \qquad \text{and } b_1 + b_2 = 1.$$

(i) Construct the Butcher's table to state the above scheme.

(ii) Letting $b_1 = 0$ construct the second order Runge-Kutta scheme.

(iii) Apply the second order Runge Kutta scheme you constructed to approximate $y(0.1)$ of the initial value problem

$$y' = ty, \quad y(0) = 1$$

with the step-size $h = 0.1$.

(iv) Compare your result with the exact solution.

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3. a) Define the following operators on a real valued function f.

(i) Shift operator (E),

(ii) Forward difference operator (Δ),

(iii) Average operator (μ).

b) Prove that, in the usual notation,

(i) $\Delta + 1 = E$,

(ii) $2\mu(1 + \Delta)^{1/2} = 2 + \Delta$.

c) Solve the difference equation $y_{n+2} - 6y_{n+1} + 9y_n = 0$, subject to $y_0 = 1, y_1 = 0$.

d) Consider the system of differential equations

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \text{where the matrix } A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(i) Show that an eigen vector $\mathbf{v}$ of A can be expressed as

$$\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \quad \text{where } i = \sqrt{-1}.$$

(ii) Use the result in part d)(i) to find the general solution of the above system of differential equations.

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4. a) (i) Using the conservation law with necessary assumptions derive the one dimensional traffic flow model,in the usual notation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0,$$

where $\rho(x,t)$ and $v(x,t)$ are the density and the speed of vehicles respectively.

(ii) Assume that $v = v(\rho)$ and v is given by the *Underwoods Exponential Model*

$$v = v_{max} e^{-\frac{\rho}{\rho_{max}}},$$

where v_{max} and ρ_{max} are maximum density and the maximum speed respectively. Describe this relation graphically and explain why this model is not suitable to predict the velocity in high densities.

(iii) Using the model in part a)(ii) above obtain the corresponding traffic flow model given in a)(i).

b) Using the method of characteristics show that the analytical solution of the initial value problem

$$u_t + xu_x = e^t, \quad u(x,0) = 1 + x$$

is given by

$$u(x,t) = e^t + xe^{-t}.$$

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5. a) Find the regions in $xy-$plane where the partial differential equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic or parabolic.

b) A bar of has the temperature profile $f(x) > 0$ for and ends are kept at temperature $0^o\ C$.

(i) State what will happen to the temperature profile of the bar as the time $t \longrightarrow \infty$.

(ii) If the initial temperature profile $f(x)$ is a linear combination of $\sin \pi x$, and $\sin 3\pi x$ of the form $f(x) = a \sin \pi x + bx \sin 3\pi x$, and satisfies the conditions

$f'(0) = f'(1) = 0$ and $f\left(\frac{1}{2}\right) = 2$, determine the constants a and b and hence obtain $f(x)$.

(iii) Write down the corresponding initial boundary value problem for the temperature $u(x,t)$ of the bar assuming the heat conductivity as 1.

(iv) Using the technique of separation of variables show that the temperature $u(x,t)$ of the rod is given by

$$u(x,t) = \frac{3}{2}\sin(\pi x)e^{-\pi^2 t} - \frac{1}{2}\sin(3\pi x)e^{-9\pi^2 t}.$$

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6. The acoustic pressure $u(x,t)$ in an organ pipe obeys the 1-D wave equation

$$u_{tt} = c^2 u_{xx}; \quad 0 \leq x \leq L, \quad t > 0,$$

where c is the speed of the sound in Air and L is the length of the pipe. Each organ pipe is closed at one end and open at the other end. At the closed end, the boundary condition is $u_x(0,t) = 0$ while at the open end $u(L,t) = 0$.

a) Given initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$,for $0 \leq x \leq L$, write down the corresponding initial boundary value problem for the acoustic pressure $u(x,t)$.

b) Let $L = 1$. Using the technique of separation of variables and boundary conditions given above, show that the general solution of the problem can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{2n-1}{2}\right)\pi ct + F_n \sin\left(\frac{2n-1}{2}\right)\pi ct \right] \cos\left(\frac{2n-1}{2}\right)\pi x,$$

where E_n and F_n are constants to be determined using initial conditions.

c) Suppose that the initial condition $g(x) = 0$. Show that the solution can be reduced to

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{2n-1}{2}\right)\pi ct \right] \cos\left(\frac{2n-1}{2}\right)\pi x.$$

d) Suppose that $f(x) = 1,\ 0 \leq x \leq 1$. Explain briefly how you would obtain the final solution of the problem determining unknown coefficients. (No need to evaluate integrals (if any)).
