UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: December 2018

Module Number: IS2401 **Module Name: Linear Algebra & Differential Equations**

[Three hours]

[Answer **all questions**, each question carries **12** marks]

Q1. a) Suppose that in winter the day time temperature in a certain office building is maintained at 70 °F. The heating is shut off at 10 p.m. and turned on again at 6 a.m. On a certain day the temperature inside the building at 2 a.m. was found to be 65 °F. The outside temperature was 50 °F at 10 a.m. and had dropped to 40 °F by 6 a.m. What was the temperature inside the building when the heat was turned on at 6 a.m.?
You may assume that, if $T(t)$ is the temperature inside the building and T_A the outside temperature (Assumed to be constant in Newton's law), then the time rate of change of the temperature T is propotional to the difference between T and T_A are given by

$$\frac{dT}{dt} = k(T - T_A),$$

and T_A is the average temperature varying from 50 °F and 40 °F.

[3 Marks]

b) Find an integrating factor of $(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$ and hence, solve the problem when $y(0) = -1$.

[2 Marks]

c) What is meant by
   i. an ordinary point
   ii. a regular singular point of a differential equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x)$$

[2 Marks]

d) i. Show that infinity is a regular singular point of the differential equation

$$2x^2(x-1)\frac{d^2y}{dx^2} + 3x(x-1)\frac{dy}{dx} + 3y = 0$$

ii. Solve the above differerntial equation about the infinity.

[5 Marks]

Q2. a) i. Define an irrotational field.

ii. Show that $\mathbf{F} = y(z-2x)\mathbf{i} - x(x-z)\mathbf{j} + xy\mathbf{k}$ is an irrotational field and hence, find the scalar potential of $\mathbf{F}$.

iii. Suppose particle of constant mass m is moving in a conservative force field $\mathbf{F} = \nabla\phi$. If A and B are any two points in space and v_A and v_B are the magnitudes of the velocities of the particle at A and B respectively, prove that

$$\phi(A) - \phi(B) = \frac{1}{2}m(v_A^2 - v_B^2)$$

[5 Marks]

b) If $\mathbf{F} = (2y+3)\mathbf{i} + xz\mathbf{j} + (yz-x)\mathbf{k}$, evaluate $\int_C \mathbf{F}.d\mathbf{r}$ along each of the following path C.

i. $x = 2t^2, y = t, z = t^2$ from $t = 0$ to $t = 1$.

ii. The straight lines from (0,0,0) to (0,0,1) then to (0,1,1) and then to (2,1,1).

[4 Marks]

c) If $\mathbf{A} = (x-y)\mathbf{i} - xz\mathbf{j} + 2y\mathbf{k}$, evaluate $\int_V \nabla.\mathbf{A}\ dv$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $x + 2y + 2z = 4$.

[3 Marks]

Q3. a) State

i. The divergence theorem

ii. Stockes' theorem.

[2 Marks]

b) Let $\mathbf{V}$ be a volume bounded by a closed surface S. Prove that

$$\iiint_V \nabla\phi\ dV = \iint_S \phi\mathbf{n}\ dS,$$

where ϕ is a scalar function and $\mathbf{n}$ is the outword unit normal vector.

[3 Marks]

c) If S is the surface $x^2 + y^2 + z^2 = 4$ and $\mathbf{A} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$, use divergence theorem to evaluate $\iint \mathbf{A}.\mathbf{n}\ dS$.

[3 Marks]

d) If $\mathbf{B} = 2(x+1)\mathbf{i} - z\mathbf{j} + (y+z)\mathbf{k}$, use stockes' theorem to evaluate $\iint_S (\nabla \times \mathbf{B})dS$, where S is the surface of the cylinder which is open from the top and bounded by the surfaces $x^2 + y^2 = 1$ and xy plane.
Note that $\mathbf{n}$ be the outward unit normal to the given surface.

[4 Marks]

Q4. a) Let V be a vector space over the field F. Explain what is meant by

i. S is a linearly independent set of V.

ii. S a spanning set of V.

[2 Marks]

b) State whether each of the following is true or false. Justify your answers.

i. $W = \{(x,y); x,y,z \in \mathbb{R},\ x+y=0\}$ is a subspace of $\mathbb{R}^2$ under the usual addition and scalar multiplication.

ii. If $U = \{(x,0,z); x,z \in \mathbb{R}\}$ and $W = \{(x,y,0); x,y \in \mathbb{R}\}$, $U \cup W$ is a subspace of V.

iii. $S = \{(1,0,2,1),(1,-1,1,0),(2,-1,3,1),(1,1,3,2)\}$ is a basis for $\mathbb{R}^4$.

iv. $S = \{(1,0),(0,1),(1,1)\}$ is a spanning set of $\mathbb{R}^2$

[6 Marks]

c) Let U and W be subspaces of $\mathbb{R}^4$ generated by the following two sets S and T respectively, here,

$$S = \{(2,1,-1,2),(1,0,1,0),(1,1,-2,2),(0,-1,3,-2)\} \text{ and}$$
$$T = \{(1,-1,0,-2),(1,2,-1,4),(1,-2,3,-4),(0,1,-3,2)\}$$

Find the dimensions of U, W, $U+W$ and $U \cap W$.

[4 Marks]

Q5. a) Briefly explain

i. Image and Kernel

ii. Rank and Nullity

of a linear transformation.

[2 Marks]

b) Let $T_A: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation where,

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ -2 & 1 & -1 & 2 \\ 1 & 2 & -1 & -1 \end{bmatrix}$$

Find bases and dimensions of the Kernel and the Image of T_A.

Verify that

$$\dim(\mathbb{R}^4) = \text{rank}(T_A) + \text{nullity}(T_A)$$

[4 Marks]

c) i. Find the eigenvalues and eigenvectors of the following matrix A.

$$A = \begin{bmatrix} 2 & -3 & 3 \\ -3 & -2 & -3 \\ -3 & 3 & -4 \end{bmatrix}$$

ii. Diagonalize the above matrix A.

[6 Marks]