UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: March 2021

Module Number: IS3302 **Module Name: Complex Analysis and Mathematical Transforms (C 18)**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

Q1. a) Discuss the differentiability of the function $f(z) = z^n \; ; n \epsilon \mathbb{N}$.

[2 Marks]

b) Discuss the continuity of the following functions

i $f(z) = \begin{cases} \frac{Re(z^2)}{|z|} & ; \; z \neq 0 \\ 0 & ; \; z = 0 \end{cases}$ at $z = 0$.

ii $f(z) = \begin{cases} \frac{z^2+1}{z-i} & ; \; z \neq i \\ 0 & ; \; z = i \end{cases}$ at $z = i$.

[4 Marks]

c) In the usual notations, z and w are two complex numbers in Z and W planes respectively. Find the image of the circle $|z| = 2$ under the transformation $w = iz + 1$.

[3 Marks]

d) Consider the conformal transformation $w = z^2$.

i Find the coefficient of magnification at $z = 2 + i$.

ii Find the angle of relation at $z = 2 + i$.

[3 Marks]

Q2. a) Show that $f'(z) = \frac{\partial u}{\partial x}(z, 0) - i\frac{\partial u}{\partial y}(z, 0)$. Then use it to find the harmonic conjugate of $u(x, y) = e^{-x}(x \sin y - y \cos y)$.

[3 Marks]

b) Find the Maclaurin expansion of $f(z) = \cos z$. Hence, write down the expansion of $\sin^2 z$ to powers of x^6.

[4 Marks]

c) Use Cauchy's Residue Theorem to evaluate the integral $\oint_C \frac{1}{z^2(z-2)(z-4)} dz$ if C is the rectangle joining the points $(-1, -1), (3, -1), (3,1)$ and $(-1,1)$.

[5 Marks]

Q3. a) In the usual notations, if the Laplace transform of the function $f(t)$: $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$.

Hence, find the Laplace transform of the followings.

i $t^2u(t-3)$

ii $e^{-2t}u(t-\pi)$

[5 Marks]

b) Use Laplace transform to solve the systems of differential equations with the specified initial conditions given below.

$$\frac{d^2x}{dt^2} = y + \sin t \quad ; \; x(0) = 1, x'(0) = 0$$
$$\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \cos t \quad ; \; y(0) = -1, y'(0) = -1$$

[4 Marks]

c) Use Convolution theorem to find the inverse Laplace transform of $\frac{1}{s^2(s+1)}$.

[3 Marks]

Q4. a) In the usual notations of Z transform, show that $Z\big(x(n-m)\big) = z^{-m}X(z)$; where m is a positive integer and $x(n)$ is a causal sequence.
Hence, find the Z transform of $y(n)$.
Where
$y(n) = (0.5)^{n-5}u(n-5)$ and $u(n-5) = \begin{cases} 1 & n \geq 5 \\ 0 & n < 5 \end{cases}$

[4 Marks]

b) Use Z transform to solve the difference equation;
$y_{n+2} = y_{n+1} + y_n$; $n = 0, 1, 2, \ldots$; $y(0) = y(1) = 1$.

[4 Marks]

c) If $x_1(n)$ and $x_2(n)$ are two sequences, then find the Z transform of their convolution.
Where, $x_1(n) = 4\delta_n + 3\delta_{n-1}$; $x_2(n) = \delta_n - 2\delta_{n-1}$.

[4 Marks]

Q5. a) Consider the Fourier Series for a function $f(t)$ of period 2π ;
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$
Where,
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$; $n = 1,2,3,\ldots$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$; $n = 1,2,3,\ldots$

i Find the Fourier series expansion of the function $f(t) = \frac{t}{4}$ over the interval $0 < t < 2\pi$ and has period 2π.

ii Use the Fourier series for $f(t) = \frac{t}{4}$ in $0 < t < 2\pi$ to show that
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

[6 Marks]

b) In the usual notations, equations of the Fourier transform and inverse Fourier transform are
$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ and $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$ respectively.

i If $f(t)$ is a real even function, then find the real and imaginary parts of the Fourier transform of $f(t)$.

ii Use the integral definition to find the Fourier transform of the exponential function
$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t \geq 0 \end{cases}$; where α is a positive constant.

iii Find the inverse Fourier transform of $\frac{6\sin 2\omega}{\omega}e^{-4i\omega}$.
$\left[Hint\text{: } F\{p_a(t)\} = \frac{2a\sin \omega a}{\omega a}\right]$

[6 Marks]