# UNIVERSITY OF RUHUNA

## Faculty of Engineering

End-Semester 1 Examination in Engineering: July 2016

**Module Number: IS1401** **Module Name: Mathematical Fundamentals for Engineers**

**[Three hours]**

[Answer **all questions,** each question carries fourteen marks]

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Q1. a) Find all the values of $z \in \mathbb{C}$, and locate them on the Argand diagram if,

$$\cosh(3\ln z) + \sinh(3\ln z) + 1 = 0.$$

[4 Marks]

b) i. If $n \geq 2$, prove that summation of all $n^{\text{th}}$ roots of unity is zero.

ii. If $\omega$ is a cubic complex root of unity, without computing $\omega$, show that the product of the following two matrices $A$ and $B$ is a zero matrix.

$$A = \begin{pmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

[5 Marks]

c) Use De Moivre's theorem to express $\cos 6\theta$ in terms of $\cos\theta$ and $\sin\theta$. Hence, solve the equation

$$16x^6 - 24x^4 + 9x^2 = 0$$

[5 Marks]

Q2. a) If $A$ and $B$ are square matrices, show that

i. $(AB)^{-1} = B^{-1}A^{-1}$

ii. $(A^{-1})^T = (A^T)^{-1}$

[2 Marks]

b) Find the inverse of the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & -a & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & b & b^2 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, find the inverse of

$$\begin{pmatrix} 1 & b & b^2 \\ a & ab+1 & ab^2 - b \\ a^2 & a^2b - a & a^2b^2 + ab + 1 \end{pmatrix}$$

and solve the system of linear equations

$$\begin{aligned} x - 2y + 4z &= 7 \\ -x + 3y - 2z &= -6 \\ x - y + 7z &= 9 \end{aligned}$$

[8 Marks]

c) Find the values of $\lambda$ and $\mu$ such that the following system has
   i. unique solution.
   ii. infinitely many solutions.
   iii. no solution.

$$\begin{aligned} x + 2y - z &= 2 \\ 2x - y + 2z &= \mu \\ x + \lambda y + 3z &= 1 \end{aligned}$$

[4 Marks]

Q3. a) i. Briefly explain 'position vector' and 'unit vector'.
ii. If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ are position vectors of points $A$ and $B$ respectively, find the vector $\overline{AB}$.
Show that $OAB$ is a rectangular triangle.

[3 Marks]

b) i. Define 'Triple vector product' and 'Triple scalar product' of three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.
ii. Use triple scalar product to show that vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are co-planar, when $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

[4 Marks]

c) i. Prove that $\nabla\phi$ is perpendicular to the surface $\phi(x, y, z) = C$. Where $C$ is a constant.
ii. If $\phi = x^2yz - 2xy^2$, find a unit vector perpendicular vector to $\phi$ at the point $(1,2,-1)$.

Hence, find the directional derivative of $\phi$ at the point $(1,2,-1)$ in the direction of $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

iii. Show that

$$\nabla\left(\frac{\mathbf{r}}{r^2}\right) = \frac{1}{r^2}$$

Where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of any arbitrary point $(x, y, z)$ and $r = |\mathbf{r}|$.

[7 Marks]

Q4. a) Write down an example for function $f(x)$, which satisfies each of the following condition.

i. $f(x)$ is continuous for all $x \in \mathbb{R}$, but not differentiable at $x = 2$.
ii. Limit of $f(x)$ exists for all $x \in \mathbb{R}$, but $f(x)$ is not continuous for any $x \in \mathbb{Z}$
iii. Limit of $f(x)$ does not exist for any $x \in \mathbb{R}$
iv. Limit of $f(x)$ exists only for $x \in \mathbb{R} - \mathbb{Z}$

[4 Marks]

b) Evaluate the following limits

i. $\lim\limits_{x \to 0} \dfrac{x(\cos^3 x - 1)}{\sin^3 x}$  ii. $\lim\limits_{x \to \infty} \dfrac{\sqrt{(x^2 - 1)}}{x}$

[4 Marks]

c) Discuss the continuity of the following functions.

$$f(x) = \begin{cases} 2x+1\,; & x < -2 \\ x-1\,; & -2 \le x < 0 \\ x+1\,; & 0 \le x < 1 \\ 2x\,; & 1 \le x \end{cases}$$

[2 Marks]

d) Sketch the graphs of

i. $y = |x+1| - |2x-1| + |x-2|; \quad x \in \mathbb{R}$

ii. $y = [x^2+1]; \quad x \in \mathbb{R}$

[4 Marks]

Q5. a) i. State the Rolle's theorem.

ii. State and prove the 'Mean Value Theorem'.

[4 Marks]

b) If $f(x)$ and $g(x)$ are continuous and differentiable functions on $\mathbb{R}$ and $f(a) = f(b) = 0$, show that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

Hence, find the limit of

$$\lim_{x\to -1}\frac{\ln(x+2)}{x^2-1}$$

[5 Marks]

c) i. If $z = f(x,y)$ is a continuous and differentiable functions on $\mathbb{R}$, show that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

When $x$ and $y$ are functions of variable $u$, deduce that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

ii. If $f(x,y)$ is a function of variables $x$ and $y$, where $x = u\cos\theta - v\sin\theta$ and $y = u\sin\theta + v\cos\theta$ ; $\theta$ being a constant, show that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

[5 Marks]