UNIVERSITY OF RUHUNA
BACHELOR OF SCIENCE SPECIAL DEGREE(LEVEL I/II) SEMESTER II EXAMINATION JULY - 2020

PHY4034 - Quantum Mechanics Part II **Time: 02 Hours and 30 minutes**

Answer Five Questions Only

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

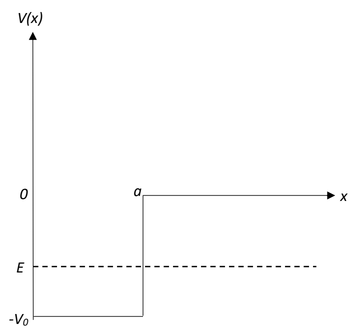
---

1. Schrödinger equation for a given potential can be written as $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. If the potential is independent of time (t), wave function $\Psi(x,t)$ can be written as $\Psi(x,t) = \psi(x)\phi(t)$

   (a) i. Find the solution to time dependent wave function $\phi(t)$ where E is the separation constant.

   ii. Derive the time independent Schrödinger equation.

   (b) A particle of mass m is moving from $-x$ direction towards $+x$ direction in the following potential.



$$V(x)=\begin{cases}\infty, & x \leq 0\\ V_0, & 0 < x < a\\ 0, & x \geq a\end{cases}$$

where $V_0 > 0$.

Suppose $-V_0 < E < 0$, where E is the energy of the particle. let $k_1 = \sqrt{\frac{2m\,(V_0+E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{-2mE}{\hbar^2}}$

   i. Find the time independent wave functions for the regions where $V(x) = \infty$, V_0 and 0 respectively.

   ii. Using the boundary conditions at $x = 0$ and $x = a$, show that the formula for the allowed energies can be written as $k_1 a \cot(k_1 a) = -k_2 a$.

   iii. Use a graphical method and show that $V_0 > \frac{\pi^2\hbar^2}{8ma^2}$ in order to have at least one bound state.

2. (a) Three states of a system are defined as $|\alpha\rangle = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$,

$|\beta\rangle = \sqrt{\frac{1}{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$, and $|\gamma\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Write down the corresponding bra vectors.

(b) Consider a physical system defined by the Hamiltonian, $H = \epsilon \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *and state* $\psi_0 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$

i. What values will we obtain when measuring energy of the system?
ii. How many degenerate states you would have in the system?
iii. Find the eigen vecotrs of H and show that they are equal to $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle$ in part (a).
iv. Write down the state ψ_0 as a linear combination of the eigenvectors of H?
v. What are the probabilities of finding each energy eigenvalue?
vi. Calculate expectation value of H ($< H >$).

3. Hamiltonian of a harmonic oscillator is given as $\hat{H} = \frac{1}{2}(\frac{\hat{P}^2}{m} + mw^2\hat{x}^2)$ where $\hat{P}$ and $\hat{x}$ are momentum and position operators. Raising and lowering operators of a harmonic oscillator are defined as $\hat{a}_{\pm} = \frac{1}{\sqrt{2\hbar\omega m}}(\mp i\hat{P} + m\omega\hat{x})$.

(a) i. Show that $\hat{H}$ can be written as $\hbar\omega(\hat{a}_{\mp}\hat{a}_{\pm} \mp \frac{1}{2})$.
ii. Show that the commutation relation, $[\hat{a}_-, \hat{a}_+] = 1$.
iii. Write down the operators $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}_+$ and $\hat{a}_-$.
iv. Find the expectation value of the total energy in the n^{th} state of the harmonic oscillator.

(b) Consider a particle of mass m and charge q moving under the influence of a one dimensional harmonic oscillator potential. The particle is placed in a constant electric field ϵ and the Hamiltonian of the particle is given by,
$\hat{H}_0 = \frac{1}{2}(\frac{\hat{P}^2}{m} + mw^2\hat{x}^2) - q\epsilon\hat{x}$.
i. Show that the H_0 can be written in the form, $\hat{H}_0 = \hat{H} - K$ where K is a constant that depends on ϵ, q, m and ω.
ii. Derive an expression for the energy in n^{th} excited state.
(You may use the relations, $\hat{a}_- |n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$)

4. The wave function of a hydrogen atom can be written as $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \Phi)$. Where $R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$ and the Bohr radius is given by the relation $\rho = \frac{r}{an}$.

   (a) Given that $v(\rho) = \sum_{j=0}^{+\infty} c_j \rho^j$ and $c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} c_j$ show that the radial wave function $R_{21}(r) = \frac{C_0}{4a^2} r e^{\frac{-r}{2a}}$.

   (b) Consider a spinless particle represented by a wave function,
   $\psi(r, \theta, \phi) = Ae^{-\alpha r}(cos\,\phi\, sin\,\theta\, +\, sin\,\phi\, sin\,\theta\, +\, cos\,\theta)$

      i. Show that the angular wave function($Y(\theta, \Phi)$) can be written as a linear combination of spherical harmonics.

      ii. Find the normalization constant A.

      iii. What is the total angular momentum of the particle?

      iv. What is the expectation value of the z component of the angular momentum ($< L_z >$)?

      v. If the z component of the angular momentum was measured, what is the probability that the result would be $L_z = +\hbar$?

   The first few spherical harmonics are given as: $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\, cos\,\theta$, $Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, sin\,\theta\, e^{\pm\phi}$, $Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\, sin\,\theta\, cos\,\theta\, e^{\pm\phi}$.

   You may use the orthonormalization relation: $\int_0^{2\pi} d\Phi \int_0^{\pi} sin\ \theta\, d\theta\ Y_{l'}^{m'}(\theta, \Phi)\, Y_l^m(\theta, \Phi) = \delta_{l'l}\delta_{m'm}$

5. Assume that the Hamiltonian H of a system is given by $H = H^0 + \lambda H^1$. Where H^0 and H^1 are given as unperturbed and perturbed Hamiltonians of the system.

   (a) Write ψ_n and E_n as power series of λ and show that the 1^{st} order correction for the energy, $E_n^1 = \langle\psi_n^0|\, H^1\, |\psi_n^0\rangle$.

   (b) A spinless particle of mass m moving in an infinite one dimensional potential well of length $2L$, with $x = 0$ and $x = 2L$:

   $$V(x)=\begin{cases} 0 & 0 \le x \le 2L \\ \infty & \text{otherwise.} \end{cases}$$

      i. Use the Schrödinger equation to find the energy of the n^{th} excited state.

      ii. The systems is slightly modified at the bottom by using the the purturbation $V_p(x) = \lambda V_0 sin(\frac{\pi x}{2L})$. Assume that both λ and V_0 are constants. Calculate the 1^{st} order correction for the energy of n^{th} state of the system.

6. (a) In which situations the variational principal is useful in Quantum mechaincs?

   (b) Suppose that the ground state energy of a system described by a Hamiltonian H is E_g. Assuming that the wave function Ψ is normalized and the H forms a complete set, show that $E_g \leq \langle\Psi| H |\Psi\rangle$.

   (c) Use the variational method to estimate the ground state energy of a particle of mass m moving in one dimensional potential well $V(x) = \lambda|x|$; λ is a constant (Hint: You may use the trial function $Ae^{-\alpha|x|}$; where A is the normalization constant and α is a scale parameter.)