UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: August 2015

Module Number: IS 3301 **Module Name: Complex Analysis and Mathematical Transforms**

[Three hours]

[Answer **all questions**, each question carries fourteen marks]

Q1. a) State the Cauchy's integral formula in the usual notation.
Evaluate

$$\int_C \frac{e^z}{(z+2)(z+1)^2}dz; \quad C: |z| = 3$$

[5 Marks]

b) i) Obtain the Taylor's series expansion of $f(z) = \sin z$ upto third order derivative about the point $z = \frac{\pi}{4}$.

ii) Consider the function

$$f(z) = \frac{1}{z^2 - 3z + 2}.$$

Using partial fractions, show that the Laurent's series expansion of $f(z)$ in the region $1 < |z| < 2$ is given by

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}.$$

[9 Marks]

Q2. a) Find the image of

i) the square region with vertices $(0,0), (2,0), (2,2), (0,2)$ under the transformation $w = (1+i)z + (2+i)$.

ii) the circle $|z - 3| = 5$ under the mapping $w = \frac{1}{z}$.

Sketch the image on the w-plane in each case.
Describe the nature (that is, the translation, rotation, and expansion or contraction) of the image in part i).

[8 Marks]

b) State the Cauchy's residue theorem in the usual notation.
Evaluate

$$\int_0^{2\pi} \frac{d\theta}{2\cos\theta + 3}.$$

[6 Marks]

Q3. a) If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$, where a is a real constant.

Find $\mathcal{L}\{\sin at\}$ and using the above result and stating any other result you may use, show that

$$\mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\} = \cot^{-1}(s+1)$$

[3 Marks]

b) i) Using partial fractions, find

$$\mathcal{L}^{-1}\left\{\frac{s^2 - 10s + 13}{(s-7)(s^2 - 5s + 6)}\right\}.$$

ii) Apply the convolution theorem to show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t\sin 2t}{4}.$$

[Hint: You may use the trigonometric identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ **in integration, if necessary.]**

[5 Marks]

c) Show, in the usual notation, that

i) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

ii) $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

Hence, solve the differential equation

$$y'' + 9y = 18t,$$

given that $y(0) = 0 = y\left(\frac{\pi}{2}\right)$

[6 Marks]

Q4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x; & 0 < x < \pi \\ \pi; & \pi < x < 2\pi \end{cases}$$

with period 2π.

a) Sketch the graph of $f(x)$ in the interval $-2\pi < x < 2\pi$. What can you say about the behaviour of the function at $x = 0$?

[01 Mark]

b) Calculate the Fourier coefficients and show that the Fourier series of the given function can be written as

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

Hence, deduce that

i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

ii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

[13 Marks]

Q5. a) i) Show that the Fourier transform $F(s)$ of the function

$$f(x) = \begin{cases} x; & \text{if } |x| \leq a \\ 0; & \text{if } |x| > a \end{cases}$$

is given by

$$F(s) = \frac{i}{s^2}\sqrt{\frac{2}{\pi}}[\sin sa - as\cos sa]$$

Hence, find the Fourier transform of the function

$$f(x) = \begin{cases} x^2; & \text{if } |x| \leq a \\ 0; & \text{if } |x| > a \end{cases}$$

ii) Consider the function $f(x) = e^{-3x}\cos 3x$. Show that the Fourier cosine transform of $f(x)$, that is $F_C[f(x)]$, is given by

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{3(s^2+18)}{s^4+324}\right]$$

[Hint: You may assume that $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$]

[7 Marks]

b) Show, in the usual notation, that

$$Z[a^n] = \frac{z}{z-a}$$

where a is any real or complex number.

i) Find the inverse Z -transform of

$$F(z) = \frac{z^2+2z}{(z-1)(z^2-5z+6)}$$

by the partial fraction method.

ii) Find the solution of the difference equation

$$y(n+2) - 5y(n+1) + 6y(n) = 4^n;\ y(0) = 0 \text{ and } y(1) = 1$$

by using Z -transforms.

[7 Marks]