UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: July 2022

Module Number: IS2401 **Module Name: Linear Algebra & Differential Equations**

[Three hours]

[Answer **all questions,** each question carries **12** marks]

Q1. a) An electrical circuit has an inductance L in series with a resistance R and is connected to an alternating voltage supply of $V\cos\omega t$. The current i at a time t after switch is closed and the voltage is applied, is given by the differential equation

$$L\,\frac{di}{dt} + Ri = V\cos\omega t.$$

Solve the equation if the current is zero when $t = 0$.

[3 Marks]

b) Prove that, if $F(-\alpha^2) \neq 0$ then

i. $\frac{1}{F(D^2)}\{\cos\alpha x\} = \frac{1}{F(-\alpha^2)}\cos\alpha x$

ii. $\frac{1}{F(D^2)}\{\sin\alpha x\} = \frac{1}{F(-\alpha^2)}\sin\alpha x$

Hence determine the value of $\frac{1}{D^2-2}\sin 4x$.

[4 Marks]

c) i. Show that the following differential equation has regular singular point at infinity.

$$2x^2(x-1)\frac{d^2y}{dx^2} + 3x(x-1)\frac{dy}{dx} + 3y = 0$$

ii. Solve the above differerntial equation about the infinity.

[5 Marks]

Q2. a) i. Define a conservative vector field.

ii. Let $F = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives. Show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in the field F to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

[4 Marks]

b) i. Let $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Show that $\mathbf{F}$ is a conservative force field.

ii. Find the scalar potential such that $F = \nabla\phi$.

iii. Calculate the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

[5 Marks]

c) Evaluate $\iint_S A.n\,ds$, where $A = z\mathbf{i} + x\mathbf{j} - 3y^2z\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

[3 Marks]

Q3. a) State

i. The divergence theorem

ii. Stockes' theorem.

[3 Marks]

b) Verify the divergence theorem for $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4,\ z = 0, z = 3$.

[5 Marks]

c) Let V be the volume bounded by the closed surface S. The scalar field ϕ and φ are acting on the surface S. If $\mathbf{n}$ is the outward unit normal vector to the surface S at the point (x, y, z) and $\mathbf{r}$ is the position vector of the point (x, y, z), prove that

i. $\iiint_V (\phi\nabla^2\varphi - \varphi\nabla^2\phi)dv = \iint_S (\phi\nabla\varphi - \varphi\nabla\phi)\,dS$

ii. $\int_C \phi d\mathbf{r} = \iint_S (\mathbf{n} \times \nabla\phi)ds = \iint_S\ d\mathbf{S} \times \nabla\phi$

[4 Marks]

Q4. a) If V is a vector space over a field F then show that

i. For any scalar $k \in F$ and $\mathbf{0} \in V,\ k0 = \mathbf{0}$.

ii. For $0 \in F$ and for any $v \in V$, $0v = \mathbf{0}$

iii. If $kv = \mathbf{0}$, where $k \in F$ and $v \in V$, then $k = 0$ or $v = \mathbf{0}$

[3 Marks]

b) i. Let $V = \mathbb{R}^3$. Write down two subsapaces of V. Justify your answer.

ii. Let U and W be two subspaces of the vector space V. If $U \cup W$ is also a subspaceof V, show that $U \subseteq W$ or $W \subseteq U$.

[5 Marks]

c) Let $S = \{(-1,2,1,-4),\ (2,1,3,3), (3,-2,1,8), (1,0,1,2),\ (0,1,1,-1)\}$.

i. Determine whether S is linearly independent.

ii. Determine whether S spans $\mathbb{R}^4$.

iii. If W is a subspace generates by S, find a basis and the dimension of W.

iv. Extend the basis in iii. above to a basis of $\mathbb{R}^4$.

[4 Marks]

Q5. a) Let $T: V \rightarrow U$ be a linear transformation, where V and U are two vector spaces over the field $\mathbb{F}$.

i. Briefly explain what is meant by the Kernel and Image of T.

ii. Show that $T(0) = 0$.

iii. Show that $Ker(T)$ is a subspaces of V.

[3 Marks]

b) Consider the matrix A corresponds to the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$, where

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & -2 \\ 3 & 1 & 0 \\ 1 & -3 & 4 \end{bmatrix}.$$

Find a basis and the dimension of the kernel and the image of T.

[2 Marks]

c) If the matrix A has eigenvalue λ, show that

i. A^{-1} has eigenvalue $\frac{1}{\lambda}$

ii. A^2 has eigenvalue λ^2

[2 Marks]

d) Consider the matrix

$$A = \begin{bmatrix} 5 & -4 & 8 \\ 8 & -7 & 8 \\ 0 & 0 & -3 \end{bmatrix}$$

i. Find eigenvalues and corresponding eigen vectors of A.

ii. Show that $P^{-1}AP = D$; where D is a diagonal matrix whose entries are eigenvalues of A and P is a square matrix with corresponding eigenvectors.

iii. Hence, find A^{-1}.

iv. Write down the eigenvalues of A^{-1}.

[5 Marks]