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Higher-order local and non-local correlations for 1D strongly interacting Bose gas

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Abstract

The correlation function is an important quantity in the physics of ultracold quantum gases because it provides information about the quantum many-body wave function beyond the simple density profile. In this paper we first study the M-body local correlation functions, g_M, of the one-dimensional (1D) strongly repulsive Bose gas within the Lieb–Liniger model using the analytical method proposed by Gangardt and Shlyapnikov (2003 *Phys. Rev. Lett.* **90** 010401; 2003 *New J. Phys.* **5** 79). In the strong repulsion regime the 1D Bose gas at low temperatures is equivalent to a gas of ideal particles obeying the non-mutual generalized exclusion statistics with a statistical parameter $\alpha = 1 - 2/\gamma$, i.e. the quasimomenta of N strongly interacting bosons map to the momenta of N free fermions via $k_i \approx \alpha k_i^F$ with $i = 1, \ldots, N$. Here γ is the dimensionless interaction strength within the Lieb–Liniger model. We rigorously prove that such a statistical parameter α solely determines the sub-leading order contribution to the M-body local correlation function of the gas at strong but finite interaction strengths. We explicitly calculate the correlation functions g_M in terms of γ and α at zero, low, and intermediate temperatures. For $M = 2$ and 3 our results reproduce the known expressions for g_2 and g_3 with sub-leading terms (see for instance (Vadim *et al* 2006 *Phys. Rev.* A **73** 051604(R); Kormos *et al* 2009 *Phys. Rev. Lett.* **103** 210404; Wang *et al* 2013 *Phys. Rev.* A **87** 043634). We also express the leading order of the short distance *non-local* correlation functions $\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle$ of the strongly repulsive Bose gas in terms of the wave function of M bosons at zero collision energy and zero total momentum. Here $\Psi(x)$ is the boson annihilation operator. These general formulas of the higher-order local and non-local correlation functions of the 1D Bose gas provide new insights into the many-body physics.

1. Introduction

A fundamental principle of quantum statistical mechanics describes two types of particles: bosons which satisfy the Bose–Einstein statistics and fermions which satisfy the Fermi–Dirac statistics. An arbitrary number of identical bosons can occupy one quantum state whereas no more than one identical fermion can occupy the same quantum state. The latter fundamental concept is called the 'Pauli exclusion principle'. The statistics can be derived from the fact that the wave function of a system of bosons (fermions) is symmetric (antisymmetric) under the exchange of two particles. However, under certain conditions, a system of interacting bosons can be

mapped into another system of fermions. A significant example is Girardeau's Bose–Fermi mapping [6, 7] for the one-dimensional (1D) Lieb–Liniger Bose gas [8] with an infinitely strong repulsion, which is called the Tonks–Girardeau gas [6, 9]. This mapping was established based on the observation that for an infinitely strong repulsion the relative wave function of the interacting bosons must vanish when two bosons coincide spatially. Such behaviour mimics the Fermi statistics in identical fermions. The Bose–Fermi mapping has tremendous applications in the study of strongly interacting quantum gases of ultracold atoms [10–17] [8]. In this paper we present a new application of Girardeau's Bose–Fermi mapping to the study of higher-order local and non-local correlation functions.

An alternative description of quantum statistics is provided by Haldane's exclusion statistics [18, 19]. Haldane formulated a description of fractional statistics [18–20] based on the generalized Pauli exclusion principle, which counts the dimensions in the Hilbert space in a system with adding or removing an extra particle. It is now called the generalized exclusion statistics (GESs) [18]. In the strong coupling limit, i.e., when the interaction strength goes to infinity, the Tonks–Girardeau gas is in many ways equivalent to a non-interacting Fermi gas. In fact, the 1D δ-function interacting bosons can be mapped onto an ideal gas [20] with the GES [18] described by the statistical parameter α. The equivalence between the 1D interacting bosons and the non-interacting particles obeying the GES is in general based on the equivalence between the thermodynamic Bethe ansatz (TBA) equations [21] and the GES equation [20, 22]. The statistical profiles and the thermodynamic properties of the strongly interacting 1D Bose gas were studied through the GES and TBA approaches in [22]. On the other hand, the statistical profiles of the strongly interacting 1D Bose gas at low temperatures are equivalent to those of a gas of ideal particles obeying the non-mutual GES [22], i.e. α is independent of the quasimomenta. This equivalence has been recently investigated for a 1D model of interacting anyons [22–24]. Such an equivalence between the 1D interacting Bose gas and the ideal gas with the GES paves a way to calculating the correlation functions of the interacting system through the ideal gas. In particular, in the non-mutual GES case, we can map the quasimomenta of N strongly interacting bosons to the momenta of N free fermions via $k_i \approx \alpha k_i^F$ with $i = 1, \ldots, N$, provided that the total momentum $k_1 + \cdots + k_N = 0$. Here $\alpha = 1 - 2/\gamma$, and γ is the dimensionless interaction strength within the Lieb–Liniger model [8].

Correlation functions provide information about quantum many-body wave functions beyond the simple measurement of the density profile [25]. Therefore, the study of 2-body and M-body higher-order correlations is becoming an important theme in the physics of ultracold quantum gases [25]. The higher-order correlation was first used by Hanbury Brown and Twiss to measure the size of a distant binary star [26]. Recently the non-local M-body correlations were measured with atomic particles [27, 28]. The local pair correlation function over a wide range of coupling strengths has been determined experimentally by measuring photoassociation rates in the 1D Bose gas [30, 31]. Physically, the local pair correlation is a measure of the probability of finding two particles at the same place. Many studies have focused on the local and non-local correlations in 1D interacting uniform Bose gases at zero and finite temperatures [1–3, 25, 32–46]. Moreover, some groups have conducted the measurements of the 2-body and 3-body correlations of bosons in 1D and 3D [29, 30, 47, 48]. Recently, people have studied the dynamics of strongly interacting bosons in 3D [49, 50].

In this paper, we first calculate the higher-order correlation functions of the 1D strongly interacting Bose gas by taking the asymptotic Bethe ansatz wave function. In light of an analytical method developed by Gangardt and Shlyapnikov [1, 2], we rigorously calculate the denominator and the numerator of the M-body correlation function up to the sub-leading order. Precisely speaking, the M-particle local correlations in the strong coupling limit ($\gamma \to \infty$) can be calculated through the M-body correlation of free fermions by using Wick's theorem and the Fermi–Dirac distribution. However, for a strong but finite interaction the bosons do not exhibit pure Fermi statistics [20, 22]. It is necessary to consider a correction to the pure Fermi statistics in the Gangardt/Shlyapnikov approach. It turns out that for strong but finite interaction strengths the statistical parameter α solely determines the sub-leading order contribution to the M-body local correlation function. In the strong coupling regime, the statistical profiles and the thermodynamic properties of the 1D Bose gas are equivalent to those of the ideal gas with the GES parameter α [22]. We derive explicit formulas of g_M with sub-leading terms for arbitrary $M = 1, 2\ldots$ at zero and non-zero temperatures. For the special cases of $M = 2, 3$, our results reduce to the known results of g_2 and g_3 with the sub-leading terms as given in [1, 2, 4, 41, 51, 52]. Furthermore, we analytically calculate the leading order of the short distance M-body non-local correlation functions $\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle$ of the 1D strongly repulsive Bose gas. Here $\Psi(x)$ is the boson annihilation operator.

Our paper is organized as follows. In section 2 we derive a general formula for the M-body local correlation function g_M of 1D bosons at a large interaction strength, $\gamma \gg 1$. In section 3, we analytically calculate the M-body local correlation at various temperatures. We then compare our results for g_M with the previous results for

[8] The Bose–Fermi mapping may also be used in the reverse order. A prime example is the Usui transformation [59] which maps fermion pairs to bosons.

g_2 and g_3 by Gangardt and Shlyapnikov [2], Vadim *et al* [3], Kormos *et al* [4], and Wang *et al* [5]. In section 4 we study the wave function of M interacting bosons at zero collision energy. In section 5 we calculate the short distance M-body non-local correlation functions of the ideal Fermi gas. In section 6 we determine the short distance M-body non-local correlation functions of the 1D strongly repulsive Bose gas, expressing such correlation functions in terms of the wave functions defined in section 6. In section 7 we conclude.

2. The higher-order local correlation functions of 1D Bosons

We consider N bosons interacting via repulsive δ-function potentials in 1D with Hamiltonian

$$H = \frac{\hbar^2}{2m}\left(\sum_{j=1}^{N} - \partial_{x_j}^2 + 2c\sum_{j>l}\delta(x_j - x_l)\right), \tag{1}$$

where m is the mass of each boson, x_j is the coordinate of the jth boson and $c > 0$ is the coupling constant [8]. The Hamiltonian (1) is diagonalized by means of the Bethe ansatz [2, 8, 53]. For convenience we define the dimensionless interaction strength $\gamma = c/n$, where $n = N/L$ is the number density of the bosons. Assuming the periodic boundary condition, $\psi(0, x_2, \ldots, x_N) = \psi(x_2, \ldots, x_N, L)$, we have the energy eigenfunction [2, 8, 54]

$$\begin{aligned}\psi(x_1, x_2, \ldots, x_N) &= \left[\prod_{1\leqslant i<j\leqslant N}\left(1 + \frac{\partial_{x_j} - \partial_{x_i}}{c}\right)\right]\phi^{(0)}(x_1, x_2, \ldots, x_N)\\ &= \sum_p (-1)^p\left[\prod_{1\leqslant i<j\leqslant N}\left(1 + \frac{\mathrm{i}k_{pj} - \mathrm{i}k_{pi}}{c}\right)\right]\exp\left(\sum_{j=1}^{N}\mathrm{i}k_{pj}x_j\right)\end{aligned} \tag{2}$$

in the domain $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N \leqslant L$, where

$$\phi^{(0)}(x_1, x_2, \ldots, x_N) \equiv \sum_p (-1)^p \exp\left(\sum_{j=1}^{N}\mathrm{i}k_{pj}x_j\right), \text{ for all } x's \tag{3}$$

is a completely antisymmetric function, and $k_1, \ldots, k_N$ are the quasimomenta [8]. Without loss of generality we shall assume that $k_1 < k_2 < \cdots < k_N$. The sums in equations (2) and (3) run over $N!$ permutations of the integers $1, \ldots, N$, and $(-1)^p = +1\,(-1)$ for an even (odd) permutation. The M-particle local correlation function is defined as [2]

$$\begin{aligned}g_M &= \frac{\langle\psi|(\Psi^\dagger(0))^M(\Psi(0))^M|\psi\rangle}{\langle\psi|\psi\rangle} = \langle(\Psi^\dagger(0))^M(\Psi(0))^M\rangle\\ &= \frac{N!}{(N-M)!}\frac{\int|\psi(0,\cdots,0,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_N}{\int|\psi(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N},\end{aligned} \tag{4}$$

where $|\psi\rangle$ is the N-body energy eigenstate associated with the wave function ψ, and $\Psi^\dagger(x)$ and $\Psi(x)$ are respectively the creation and the annihilation operators of the bosons. The evaluations of the numerator and the denominator in equation (4) are extremely hard even for the strong coupling regime. In order to work out g_M we need to expand both the numerator and the denominator to the sub-leading order in the large coupling limit. After lengthy calculations, detailed in the appendix, we find

$$\begin{aligned}\int_0^L|\psi(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N &= \left[1 + \frac{2N(N-1)}{cL} + O(c^{-2})\right]\int_0^L|\phi^{(0)}(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N\\ &= [\alpha^{1-N} + O(c^{-2})]\int_0^L|\phi^{(0)}(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N,\end{aligned} \tag{5}$$

$$\begin{aligned}&\int_0^L|\psi(0,\cdots,0,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_N\\ &= c^{-M(M-1)}\int_0^L|\phi^{(\Delta)}(0,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_N + O(c^{-M(M-1)-2}),\end{aligned} \tag{6}$$

where

$$\phi^{(\Delta)}(0, x_{M+1}, \ldots, x_N) \equiv [\prod_{1\leqslant i<j\leqslant M}(\partial_{x_j} - \partial_{x_i})]\phi^{(0)}(x_1, \ldots, x_N)|_{x_1=\cdots=x_M=0}. \tag{7}$$

The quasimomenta $k_1, k_2, \ldots, k_N$ deviate from pure Fermi statistics at a large but finite interaction strength. Instead, they obey the non-mutual GES [20, 22]. The deviation from Fermi statistics for a large but finite interaction strength γ is described by the non-mutual GES parameter $\alpha = 1 - 2/\gamma$ [22]. If the total

momentum $k_1 + \cdots + k_N = 0$, we have $k_i = k_i^F \alpha + O(c^{-2})$, where $k_i^F = 2\pi m_i/L$ and the m_i's are integers satisfying $m_1 < m_2 < \cdots < m_N$.

In the strong coupling limit, $\gamma \gg 1$, making a scaling change $x_i = x_i^F/\alpha$ with $i = 1, \ldots, N$, we rewrite the numerator in equation (4) as

$$\begin{aligned}
&\int_0^L |\psi(0,\cdots,0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N \\
&= c^{-M(M-1)} \int_0^L \left| [\prod_{1\leqslant i<j\leqslant M} (\partial_{x_j} - \partial_{x_i})] \sum_p (-1)^p \exp\left(\sum_{j=1}^N \mathrm{i}k_{pj}x_j\right) \right|_{x_1=\cdots=x_M=0}^2 \\
&\times \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N + O(c^{-M(M-1)-2}) \\
&= c^{-M(M-1)} \int_0^{\alpha L} \left| [\prod_{1\leqslant i<j\leqslant M} \alpha(\partial_{x_j^F} - \partial_{x_i^F})] \sum_p (-1)^p \exp\left(\sum_{j=1}^N \mathrm{i}k_{pj}^F x_j^F\right) \right|_{x_1^F=\cdots=x_M^F=0}^2 \\
&\times \alpha^{M-N} \mathrm{d}x_{M+1}^F \cdots \mathrm{d}x_N^F \\
&+ O(c^{-M(M-1)-2}) \\
&= \frac{\alpha^{M^2-N}}{c^{M(M-1)}} \int_0^{\alpha L} |\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)|^2 \mathrm{d}x_{M+1}^F \cdots \mathrm{d}x_N^F + O(c^{-M(M-1)-2}),
\end{aligned} \tag{8}$$

where $\partial_{x_i^F}$ is the partial derivative with respect to x_i^F,

$$\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F) \equiv [\prod_{1\leqslant i<j\leqslant M} (\partial_{x_j^F} - \partial_{x_i^F})] \phi^F(x_1^F,\cdots,x_N^F)|_{x_1^F=\cdots=x_M^F=0}, \tag{9}$$

and ϕ^F is the wave function of ideal fermions:

$$\phi^F(x_1^F,\cdots,x_N^F) \equiv \sum_p (-1)^p \exp\left(\sum_{j=1}^N \mathrm{i}k_{pj}^F x_j^F\right). \tag{10}$$

It is easy to see that whenever any one of the $(N - M)$ arguments $x_{M+1}^F, \ldots, x_N^F$ goes to zero, say $x_i^F \to 0$, for some i that satisfies $M + 1 \leqslant i \leqslant N$, the function $\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)$ goes to zero like $(x_i^F)^M$. On the other hand, the function $\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)$ is periodic:

$$\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_{i-1}^F,\, x_i^F + L,\, x_{i+1}^F,\cdots,x_N^F) = \phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_{i-1}^F,\, x_i^F,\, x_{i+1}^F,\cdots,x_N^F). \tag{11}$$

Thus, whenever $\alpha L < x_i^F < L$ for some i satisfying $M + 1 \leqslant i \leqslant N$, the function $\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)$ is of the order c^{-M}. So

$$\begin{aligned}
\int_0^{\alpha L} |\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)|^2 \mathrm{d}x_{M+1}^F \cdots \mathrm{d}x_N^F &= \int_0^L |\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)|^2 \\
&\times \mathrm{d}x_{M+1}^F \cdots \mathrm{d}x_N^F + O(c^{-2M}).
\end{aligned} \tag{12}$$

Assuming that $M \geqslant 1$, we thus find

$$\begin{aligned}
\int_0^L |\psi(0,\cdots,0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N \;&=\; \frac{\alpha^{M^2-N}}{c^{M(M-1)}} \int_0^L |\phi^{(\Delta F)}(0,\, x_{M+1}^F,\cdots,x_N^F)|^2 \mathrm{d}x_{M+1}^F \cdots \mathrm{d}x_N^F \\
&+ O(c^{-M(M-1)-2}).
\end{aligned} \tag{13}$$

A brute-force calculation yields

$$\int_0^L |\phi^{(0)}(x_1,\cdots,x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N = N!\, L^N \det S, \tag{14}$$

where S is and $N \times N$ matrix with elements

$$S_{ij} = \mathrm{sinc}\frac{(k_i - k_j)L}{2}, \tag{15}$$

where

$$\mathrm{sinc}(\xi) \equiv \begin{cases} \dfrac{\sin\xi}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Since $S_{ij} = 1$ for $i = j$, and $S_{ij} = O(1/c)$ for $i \neq j$, we find

$$\det S = 1 + O(c^{-2}). \tag{16}$$

Thus

$$\int_0^L |\phi^{(0)}(x_1, \cdots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N = [1 + O(c^{-2})] N!\, L^N = [1 + O(c^{-2})] \int_0^L |\phi^F(x_1^F, \cdots, x_N^F)|^2 \mathrm{d}x_1^F \cdots \mathrm{d}x_N^F. \tag{17}$$

Substituting the above formula into equation (5), we get

$$\int_0^L |\psi(x_1, \cdots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N = [\alpha^{1-N} + O(c^{-2})] \int_0^L |\phi^F(x_1^F, \cdots, x_N^F)|^2 \mathrm{d}x_1^F \cdots \mathrm{d}x_N^F. \tag{18}$$

Substituting equations (13) and (18) into equation (4), we find

$$g_M = \frac{\alpha^{M^2-1}}{c^{M(M-1)}} \frac{N!}{(N-M)!} \frac{\int_0^L |\phi^{(\Delta F)}(0, x_{M+1}, \cdots, x_N)|^2 \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N}{\int_0^L |\phi^F(x_1, \cdots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N} + O(c^{-M(M-1)-2}). \tag{19}$$

From the definition of $\phi^{(\Delta F)}$, we find

$$\begin{aligned} &\frac{N!}{(N-M)!} \frac{\int_0^L |\phi^{(\Delta F)}(0, x_{M+1}, \cdots, x_N)|^2 \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N}{\int_0^L |\phi^F(x_1, \cdots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N} \\ &= \Delta_M(\partial_x) \Delta_M(\partial_y) \frac{N! \int_0^L \phi^{F*}(x_1, \cdots, x_N) \phi^F(y_1, \cdots, y_M, x_{M+1}, \cdots, x_N) \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N}{(N-M)! \int_0^L |\phi^F(z_1, \cdots, z_N)|^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N} \Big|_{x_1 = \cdots = x_M = y_1 = \cdots = y_M = 0}, \end{aligned} \tag{20}$$

where $\Delta_M(\partial_x) \equiv \Delta_M(\partial_{x_1}, \cdots, \partial_{x_M})$, and

$$\Delta_M(\xi_1, \cdots, \xi_M) \equiv \prod_{1 \leqslant i < j \leqslant M} (\xi_j - \xi_i) = \det \Xi \tag{21}$$

is the Vandermonde determinant. Here Ξ is an $M \times M$ matrix with matrix elements $M_{ij} = \xi_i^{j-1}$ $(1 \leqslant i, j \leqslant M)$.

Since $\phi^F(x_1, \cdots, x_N)$ is a Slater determinant, it satisfies Wick's theorem:

$$\begin{aligned} &\frac{N! \int_0^L \phi^{F*}(x_1, \cdots, x_N) \phi^F(y_1, \cdots, y_M, x_{M+1}, \cdots, x_N) \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N}{(N-M)! \int_0^L |\phi^F(z_1, \cdots, z_N)|^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N} \\ &= \sum_q (-1)^q G(x_1, y_{q1}) G(x_2, y_{q2}) \cdots G(x_M, y_{qM}), \end{aligned} \tag{22}$$

where the sum runs over all the $M!$ permutations of the integers $1, \cdots, M$, and

$$G(x, y) \equiv \frac{N \int_0^L \phi^{F*}(x, x_2, \cdots, x_N) \phi^F(y, x_2, \cdots, x_N) \mathrm{d}x_2 \cdots \mathrm{d}x_N}{\int_0^L |\phi^F(z_1, \cdots, z_N)|^2 \mathrm{d}z_1 \cdots \mathrm{d}z_N} \tag{23}$$

is the 1-particle reduced density matrix of ideal fermions. Substituting the definition of ϕ^F, we find

$$G(x, y) = \frac{1}{L} \sum_{i=1}^N \exp[-\mathrm{i} k_i^F (x - y)]. \tag{24}$$

Substituting the above formula into equation (22), and then equation (22) into equation (20), we find

$$\frac{N!}{(N-M)!} \frac{\int_0^L |\phi^{(\Delta F)}(0, x_{M+1}, \cdots, x_N)|^2 \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N}{\int_0^L |\phi^F(x_1, \cdots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N} = \frac{M!}{L^M} \sum_{i_1=1}^N \cdots \sum_{i_M=1}^N \Delta_M^2(k_{i_1}^F, \cdots, k_{i_M}^F). \tag{25}$$

Substituting this result into equation (19), we find

$$g_M = \frac{\alpha^{M^2-1} M!}{c^{M(M-1)} L^M} \sum_{i_1=1}^N \cdots \sum_{i_M=1}^N \Delta_M^2(k_{i_1}^F, \cdots, k_{i_M}^F) + O(c^{-M(M-1)-2}). \tag{26}$$

In the thermodynamic limit, the fermionic momenta $k_1^F, \cdots, k_N^F$ obey the Fermi distribution: in the interval of momenta $(k,\ k+\mathrm{d}k)$, where $\mathrm{d}k$ is large compared to $2\pi/L$ but small compared to n, the number of fermionic momenta is $\frac{L\mathrm{d}k}{2\pi}f(k)$, where

$$f(k)=\frac{1}{1+\mathrm{e}^{\frac{\hbar^2k^2/2m-\mu}{k_BT}}}, \tag{27}$$

k_B is Boltzmann's constant, and T and μ are respectively the temperature and the chemical potential of the 1D Bose gas *after it is tuned to infinite coupling adiabatically* at a fixed density n. At strong coupling, the actual temperature is very close to T. In the thermodynamic limit, we thus find

$$g_M=\frac{M!\ \alpha^{M^2-1}}{(2\pi)^M c^{M(M-1)}}\int_{-\infty}^{\infty}\mathrm{d}p_1\cdots\mathrm{d}p_M\, f(p_1)\cdots f(p_M)\Delta_M^2(p_1,\cdots,p_M)+O(c^{-M(M-1)-2}). \tag{28}$$

We would like to mention that the integral form of equation (28) with α set to 1 was derived in [2, 43]. Here the quantum statistical correction $\alpha=1-2/\gamma$ contributes to the subleading term of the high order correlation function; see the proof given in the appendix. At zero temperature $f(k)=\Theta(k_F-|k|)$, where $\Theta(\xi)$ is the Heaviside step function and the Fermi-like momentum $k_F=\pi n$ [22]. At non-zero temperatures $f(k)$ is broadened. Note also that

$$N=\int_{-\infty}^{\infty}\frac{L\mathrm{d}k}{2\pi}f(k). \tag{29}$$

Making a change of variable

$$k=2\pi nz, \tag{30}$$

we obtain

$$\frac{g_M}{n^M}=M!\left(\frac{2\pi}{\gamma}\right)^{M(M-1)}\alpha^{M^2-1}\int_{-\infty}^{\infty}\mathrm{d}z_1\cdots\mathrm{d}z_M\, N(z_1)\cdots N(z_M)\Delta_M^2(z_1,\cdots,z_M)+O(\gamma^{-M(M-1)-2}), \tag{31}$$

where

$$N(z)\equiv f(2\pi nz). \tag{32}$$

Equation (31) will be used later to calculate the higher-order local correlation functions.

3. The Higher-order local correlations at various temperatures

3.1. General considerations

From equation (29), we obtain

$$\int_{-\infty}^{\infty}N(z)\mathrm{d}z=1. \tag{33}$$

The multiple integral on the right hand side of equation (31) can be calculated by using the orthogonal polynomials in the random matrix model [55, 56], yielding

$$\int_{-\infty}^{\infty}\mathrm{d}z_1\cdots\mathrm{d}z_M\, N(z_1)\cdots N(z_M)\Delta_M^2(z_1,\cdots,z_M)=M!\prod_{j=0}^{M-1}h_j, \tag{34}$$

where h_j $(j=0,\ 1,\ 2,\ \ldots)$ are the norm-squares of the monic orthogonal polynomials $P_i(z)$ with weight function $N(z)$

$$\int_{-\infty}^{\infty}P_i(z)P_j(z)N(z)\mathrm{d}z=h_j\delta_{ij}. \tag{35}$$

The monic orthogonal polynomials can be found by using the Gram–Schmidt process

$$P_j(z)=z^j-\sum_{i=0}^{j-1}\frac{\langle z^j,\ P_i(z)\rangle}{\langle P_i(z),\ P_i(z)\rangle}P_i(z), \tag{36}$$

where $\langle A(z),\ B(z)\rangle=\int_{-a}^{b}A(z)B(z)N(z)\mathrm{d}z$ within a finite range $-\infty<a<b<\infty$. Therefore, equation (31) can be re-expressed as

$$\frac{g_M}{n^M}=(M!)^2\left(\frac{2\pi}{\gamma}\right)^{M(M-1)}\alpha^{M^2-1}\prod_{j=0}^{M-1}h_j+O(\gamma^{-M(M-1)-2}). \tag{37}$$

We shall use equation (37) to calculate the M-particle local correlation function at various temperatures.

We define

$$D_{ij} \equiv \int_{-\infty}^{\infty} z^{i+j} N(z) \mathrm{d}z \equiv D_{i+j}, \tag{38}$$

where i and j are non-negative integers. Because $N(z)$ is an even function, $D_{ij} = 0$ if $i + j$ is odd. We may expand the monic orthogonal polynomials as

$$P_i(z) = \sum_{j=0}^{i} P_{ij} z^j, \tag{39}$$

where $P_{ii} \equiv 1$. Substituting this formula into equation (35), we find

$$\sum_{k=0}^{i} \sum_{l=0}^{j} P_{ik} D_{kl} P_{jl} = h_i \delta_{ij}, \tag{40}$$

which may be written in the matrix form

$$PDP^T = h, \tag{41}$$

where h is a diagonal matrix with diagonal elements h_0, h_1, $\cdots$, and P is a lower triangular matrix whose diagonal elements are all 1. Note that in the above equation, the row number and the column number of each matrix starts from 0. Given the matrix D, we can solve equation (41) to find P_{ij} and h_i.

3.2. Zero temperature

At zero temperature, strongly interacting 1D bosons have a Fermi-like surface [22]

$$N(z) = \Theta\left(\frac{1}{2} - |z|\right). \tag{42}$$

Thus equation (35) is simplified as

$$\int_{-1/2}^{1/2} P_i(z) P_j(z) \mathrm{d}z = h_j \delta_{ij}. \tag{43}$$

We can express $P_j(z)$ in terms of the Legendre polynomials

$$Q_j(x) \equiv \sum_{k=0}^{[j/2]} \frac{(2j-2k)!(-1)^k}{2^j (j-k)! k! (j-2k)!} x^{j-2k}, \tag{44}$$

which satisfy the orthogonality condition in the interval $(-1, +1)$

$$\int_{-1}^{+1} Q_i(x) Q_j(x) \mathrm{d}x = \frac{2}{2j+1} \delta_{ij}. \tag{45}$$

Comparing the properties of $P_j(z)$ and those of $Q_j(x)$, we find

$$P_j(z) = \frac{(j!)^2}{(2j)!} Q_j(2z) \tag{46}$$

at zero temperature. Therefore

$$h_j = \int_{-1/2}^{1/2} [P_j(z)]^2 \mathrm{d}z = \frac{1}{2j+1} \left[\frac{(j!)^2}{(2j)!}\right]^2. \tag{47}$$

Substituting the above result into equation (37), we find the M-particle local correlation function at zero temperature

$$g_M = n^M \left(\frac{\pi}{\gamma}\right)^{M(M-1)} \frac{\alpha^{(M^2-1)}}{(2M-1)!!} \left[\frac{\prod_{j=1}^{M} (j!)}{\prod_{j=1}^{M-1} (2j-1)!!}\right]^2 + O(\gamma^{-M(M-1)-2}). \tag{48}$$

The above formula is accurate at the leading and sub-leading orders in $1/\gamma$.

3.3. Low temperatures, $T \ll T_d$

The Sommerfeld expansion is applied to the evaluation of the norm-squares of the monic orthogonal polynomials at low temperatures. The general expression for the moments of the distribution can be expressed as

$$D_{2j} = \frac{1}{2^{2j+1}}\left[\frac{2}{2j+1} + \frac{j}{6}\left(\frac{\tau}{\pi}\right)^2 + o(\tau^2)\right], \text{ if } T \ll T_d, \tag{49}$$

where j is any non-negative integer, $\tau \equiv T/T_d$, and $T_d \equiv \hbar^2 n^2/(2mk_B)$ is the quantum degeneracy temperature.

At low temperatures $T \ll T_d$, solving equation (41) using equation (49), we find

$$h_i = \frac{1}{2\mathrm{i}+1}\left[\frac{(\mathrm{i}!)^2}{(2\mathrm{i})!}\right]^2\left[1 + \frac{\mathrm{i}(\mathrm{i}+1)(2\mathrm{i}+1)\tau^2}{24\pi^2} + o(\tau^2)\right]. \tag{50}$$

Substituting this result into equation (37), we find

$$\frac{g^M}{n^M} = \frac{(\prod_{j=1}^{M} j!)^2\alpha^{(M^2-1)}}{[\prod_{j=1}^{M-1}(2j-1)!!]^2(2M-1)!!}\left(\frac{\pi}{\gamma}\right)^{M(M-1)} \times\left[1 + \frac{1}{48}M^2(M^2-1)\left(\frac{\tau}{\pi}\right)^2 + o(\tau^2)\right] + O(\gamma^{-M(M-1)-2}). \tag{51}$$

Equation (51) reduces to equation (48) at zero temperature. We would like to emphasize that the above explicit expression of the M-body correlation functions contains the leading and sub-leading order terms in $1/\gamma$. It is very interesting to observe the many-body correlation effects with respect to the interaction parameter γ, statistics parameter α, and the reduced temperature τ. We also see that the thermal fluctuation strongly affects the higher-order correlation functions.

3.4. High temperatures, $T_d \ll T \ll \gamma^2 T_d$

In [2] it was noticed that at temperatures $T \gg T_d$, the characteristic momentum of the particles is the thermal momentum $K_T \sim 1/\Lambda$, where $\Lambda = (2\pi\hbar^2/mk_B T)^{1/2}$ is the thermal de Broglie wavelength. Therefore, the small parameter for the expansion of the amplitudes in equation (2) is $1/\Lambda c$. Thus, it must satisfy the inequality $1/\Lambda c \ll 1$, which requires $T \ll \gamma^2 T_d$. At high temperatures $T \gg T_d$, the thermal wave length Λ is much smaller than the average distance between two particles and the system approaches a Maxwell–Boltzman distribution [22, 53]. In the Boltzmann limit $T \gg T_d$, we have $\zeta' = \mathrm{e}^{(\varepsilon(z)-\mu)/k_B T} \gg 1$, where $\varepsilon(z) \equiv \hbar^2(2\pi nz)^2/2m$, so we may simply approximate $N(z)$ as [20, 57] $N(z) \simeq \frac{1}{\zeta'} = \frac{1}{\mathrm{e}^{(\varepsilon(z)-\mu)/k_B T}}$. In the temperature regime $T_d \ll T \ll \gamma^2 T_d$, we thus have

$$D_{2j} = \frac{(2j-1)!!}{8^j\pi^{2j}}\tau^j. \tag{52}$$

Solving equation (41) using equation (52), we find

$$h_i = \frac{\mathrm{i}!\,\tau^{\mathrm{i}}}{2^{3\mathrm{i}}\pi^{2\mathrm{i}}}. \tag{53}$$

Substituting this result into equation (37), we find

$$\frac{g_M}{n^M} = (M!)^2\alpha^{M^2-1}\left(\sqrt{\frac{\tau}{2\gamma^2}}\right)^{M(M-1)}\prod_{j=0}^{M-1} j!, \quad T_d \ll T \ll \gamma^2 T_d. \tag{54}$$

3.5. Discussions

Table 1 shows the higher-order local correlations based on equations (51) and (54).

In [2], Gangardt and Shlyapnikov used the leading term of the wave function to calculate the local correlations of 1D bosons. They used Jacobi polynomials and the moments of the distribution to evaluate the left hand side of equation (34) in [2]. They calculated low-order correlation functions g_2 and g_3 in the temperature regime $T \ll T_d$. However, the coefficient of the temperature term in their g_3 disagrees with our result of g_3 in table 1. Here we have considered the fact that the $1/c$ correction to the wave function of the strongly interacting 1D bosons leads to a statistical correction to the pure Fermi statistics in their calculation of the correlation function. We were able to calculate any higher-order correlation functions in a wider range of temperatures. With the help of the GES with $\alpha = 1 - 2/\gamma$, the higher-order local correlation functions which we obtained provide sub-leading order corrections (cp. table 1). When $\alpha = 1$, correlations for the strongly interacting bosons, which correspond to the Tonks–Girardeau gas with a pure Fermi statistics, were already calculated in [2] at zero temperature. From table 1 we have

$$\frac{g_2}{n^2} = \frac{4\alpha^3}{3}\left(\frac{\pi}{\gamma}\right)^2\left[1 + \frac{1}{4}\left(\frac{\tau}{\pi}\right)^2\right], \tag{55}$$

Table 1. Higher-order correlations of bosons in two temperature regimes. The general forms of the correlation functions are given by equations (51) and (54).

$\frac{g_M}{n^M}$	$T \ll T_d$	$T_d \ll T \ll \gamma^2 T_d$
$\frac{g_2}{n^2}$	$\frac{4\alpha^3}{3}\left(\frac{\pi}{\gamma}\right)^2\left[1+\frac{1}{4}\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{2\tau}{\gamma^2}$
$\frac{g_3}{n^3}$	$\frac{16\alpha^8}{15}\left(\frac{\pi}{\gamma}\right)^6\left[1+\frac{3}{2}\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{9\tau^3}{\gamma^6}$
$\frac{g_4}{n^4}$	$\frac{1024\alpha^{15}}{2625}\left(\frac{\pi}{\gamma}\right)^{12}\left[1+5\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{108\tau^6}{\gamma^{12}}$
$\frac{g_5}{n^5}$	$\frac{65536\alpha^{24}}{1157625}\left(\frac{\pi}{\gamma}\right)^{20}\left[1+\frac{25}{2}\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{4050\tau^{10}}{\gamma^{20}}$
$\frac{g_6}{n^6}$	$\frac{16777216\alpha^{35}}{5615638875}\left(\frac{\pi}{\gamma}\right)^{30}\left[1+\frac{105}{4}\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{546750\tau^{15}}{\gamma^{30}}$
$\frac{g_7}{n^7}$	$\frac{4294967296\alpha^{48}}{79500599553375}\left(\frac{\pi}{\gamma}\right)^{42}\left[1+49\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{602791875\tau^{21}}{2\gamma^{42}}$
$\frac{g_8}{n^8}$	$\frac{70368744177664\alpha^{63}}{219470547636040325625}\left(\frac{\pi}{\gamma}\right)^{56}\left[1+84\left(\frac{\tau}{\pi}\right)^2\right]$	$\frac{759517762500\tau^{28}}{\gamma^{56}}$

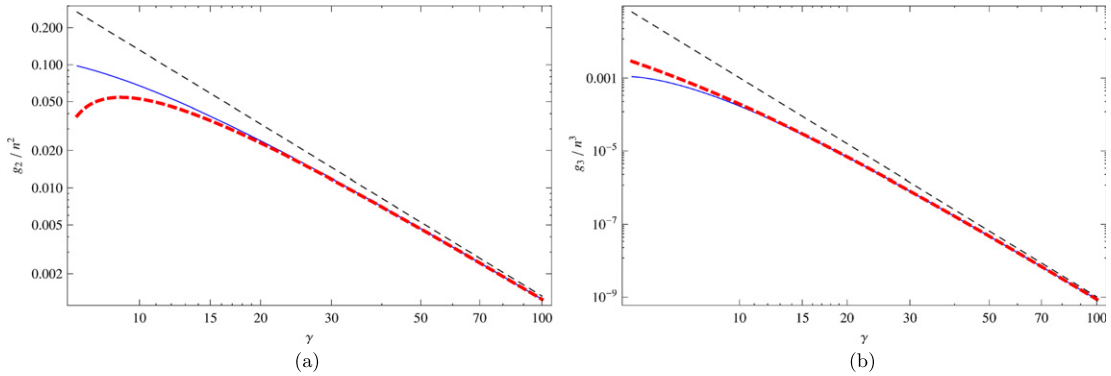

Figure 1. Double-logarithmic plots of the 2-body and 3-body local correlation functions at zero temperature. These correlation functions measure the probabilities of finding two and three particles at the same place, respectively. (a) A good agreement between our result (55) (thin blue solid line) and the exact Bethe ansatz result (57) (thick red dashed line) is observed. The thin black dashed line is the leading order of g_2 given in [2]. (b) An excellent agreement between our result (55) (thin blue solid line) and the approximate result (60) (thick red dashed line) is observed. The thin black dashed line is the leading order of g_3 given in [2].

$$\frac{g_3}{n^3} = \frac{16\alpha^8}{15}\left(\frac{\pi}{\gamma}\right)^6\left[1+\frac{3}{2}\left(\frac{\tau}{\pi}\right)^2\right]. \tag{56}$$

Besides the statistical parameter α corrections, explicit formulas of the higher-order correlation functions are presented in table 1.

In addition, Wang *et al* [5] have analytically obtained the finite temperature local pair correlations for the strong coupling Bose gas at quantum criticality using the polylog function in the framework of the TBA equations. In the Luttinger liquid phase, their result [5] reduces to

$$\frac{g_2}{n^2} = \frac{4}{3}\left(\frac{\pi}{\gamma}\right)^2\left[1-\frac{6}{\gamma}+\frac{T^2}{4\pi^2(\hbar^2n^2/2m)^2}\right], \tag{57}$$

which coincides with our result (55). Figure 1(a) shows comparisons of the 2-body correlations represented in equation (34) in [2] and the results of equations (55) and (57). Although it is clear that (55) and (57) agree with each other very well, the equation (34) in [2] has a deviation from them at strong but finite interaction strengths.

Moreover, Kormos *et al* [4, 41] developed a different method to compute the local correlation functions using the Sinh-Gordon model. They mapped the Lieb-Linger Bose gas onto the Sinh-Gordon model with certain parameter limits. From the M-particle form factor of the local operator of the Sinh-Gordon model, they

obtained the explicit form of g_2 and g_3 at $T = 0$, namely

$$\frac{g_2}{n^2} = \frac{4}{3}\left(\frac{\pi}{\gamma}\right)^2\left[1 - \frac{6}{\gamma} + \frac{1}{\gamma^2}\left(24 - \frac{8}{5}\pi^2\right)\right] + O(\gamma^{-5}), \tag{58}$$

$$\frac{g_3}{n^3} = \frac{16}{15}\left(\frac{\pi}{\gamma}\right)^6\left(1 - \frac{16}{\gamma}\right) + O(\gamma^{-8}). \tag{59}$$

The former coincides with the result given in [46], whereas the latter is consistent with our result (56) with $T = 0$. To our best knowledge, our general formula (51) derived here coincides with the known results of g_2 and g_3 [1–3, 25, 32–46]. However, there is no explicit analytical expression of the local correlation functions g_M for $M > 3$ for the 1D Bose gas in the literature. See also a recent study [43]. Cheianov *et al* obtained two approximate formulas for g_3 at medium-to-strong couplings [3]

$$\frac{g_3}{n^3}\,|_{N=\infty} = \begin{cases} \frac{0.705 - 0.107\gamma + 5.08 \times 10^{-3}\gamma^2}{1 + 3.41\gamma + 0.903\gamma^2 + 0.495\gamma^3}, & 1 \leqslant \gamma \leqslant 7, \\ \frac{16\pi^6}{15\gamma^6}\frac{9.43 - 5.40\gamma + \gamma^2}{89.32 + 10.19\gamma + \gamma^2}, & 7 \leqslant \gamma \leqslant 30. \end{cases} \tag{60}$$

Figure 1(b) compares the results among (56) and (60) and the equation (35) in [2]. It clearly shows that when $\gamma \geqslant 7$ there is a very good agreement between our result (56) and the approximate expression of Cheianov *et al* (60). All three results of g_3 reach the same asymptote in the limit $\gamma \to \infty$.

4. *M*-body wave function at zero collision energy

In this section, we consider the wave function of the 1D interacting bosons at zero collision energy and zero total momentum. If one boson has zero total momentum, its wave function is proportional to

$$\phi^{(1)}(x_1) \equiv 1. \tag{61}$$

When two bosons collide with zero total momentum and zero energy, their wave function is proportional to

$$\phi^{(2)}(x_1, x_2) \equiv |x_1 - x_2| + A, \tag{62}$$

where

$$A = \frac{2}{c}. \tag{63}$$

$(-A)$ is the so-called 1D scattering length.

If M bosons collide with zero total momentum and zero energy, and if we restrict our attention to the case in which their wave function grows no faster than $\rho^{M(M-1)/2}$ at large ρ, where ρ is the overall size of the system of M bosons, then their wave function is uniquely determined up to a multiplicative constant. This wave function is proportional to $\phi^{(M)}(x_1, x_2, \cdots, x_M)$. When $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_M$, for our convenience we define

$$\phi^{(M)}(x_1, x_2, \cdots, x_M) \equiv \lim_{k\to 0} \frac{1!\,2!\,\cdots(M-1)!}{\prod_{i<j}(\mathrm{i}k_j - \mathrm{i}k_i)} \sum_p (-1)^p \left[\prod_{i<j}\left(1 - A\frac{\mathrm{i}(k_{pi} - k_{pj})}{2}\right)\right] \times \exp(\mathrm{i}k_{p1}x_1 + \cdots + \mathrm{i}k_{pM}x_M), \tag{64}$$

where the limit is understood as follows: hold the ratio $k_1 : \cdots : k_M$ constant, and let $k_1, \ldots, k_M$ shrink to zero simultaneously. p is one of the $M!$ permutations, and $(-1)^p$ is the signature of the permutation. $(-1)^p = +1$ for even permutations and $(-1)^p = -1$ for odd permutations. In addition, we define $\phi^{(M)}(x_1, x_2, \cdots, x_M)$ to be completely symmetric under the exchanges of its arguments.

The few-body asymptotic Bethe ansatz wave functions defined here will help us to understand important correlation effects of the many-body systems.

We have the following explicit formulas at $x_1 \leqslant x_2 \leqslant \cdots$:

$$\phi^{(1)}(x_1) = 1, \tag{65}$$

$$\phi^{(2)}(x_1, x_2) = (x_2 - x_1) + A, \tag{66}$$

$$\phi^{(3)}(x_1, x_2, x_3) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) + A[(x_2 - x_1)^2 + 4(x_2 - x_1)(x_3 - x_2) + (x_3 - x_2)^2] + 3A^2(x_3 - x_1) + \frac{3}{2}A^3, \tag{67}$$

$$\begin{aligned}\phi^{(4)}(x_1, x_2, x_3, x_4) = {} & (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3) \\ & + A(\delta_2\delta_3^4 + \delta_4\delta_3^4 + 2\delta_2^2\delta_3^3 + 2\delta_4^2\delta_3^3 + 12\delta_2\delta_4\delta_3^3 + \delta_2^3\delta_3^2 + \delta_4^3\delta_3^2 + 15\delta_2\delta_4^2\delta_3^2 \\ & + 15\delta_2^2\delta_4\delta_3^2 + 4\delta_2\delta_4^3\delta_3 + 12\delta_2^2\delta_4^2\delta_3 + 4\delta_2^3\delta_4\delta_3 + \delta_2^2\delta_4^3 + \delta_2^3\delta_4^2) \\ & + A^2(\delta_3^4 + 10\delta_2\delta_3^3 + 10\delta_4\delta_3^3 + 12\delta_2^2\delta_3^2 + 12\delta_4^2\delta_3^2 + 48\delta_2\delta_4\delta_3^2 + 3\delta_2^3\delta_3 \\ & + 3\delta_4^3\delta_3 + 33\delta_2\delta_4^2\delta_3 + 33\delta_2^2\delta_4\delta_3 + 3\delta_2\delta_4^3 + 9\delta_2^2\delta_4^2 + 3\delta_2^3\delta_4) \\ & + \frac{1}{2}A^3(3\delta_2^3 + 42\delta_3\delta_2^2 + 39\delta_4\delta_2^2 + 66\delta_3^2\delta_2 + 39\delta_4^2\delta_2 + 144\delta_3\delta_4\delta_2 + 16\delta_3^3 \\ & + 3\delta_4^3 + 42\delta_3\delta_4^2 + 66\delta_3^2\delta_4) \\ & + 3A^4(3\delta_2^2 + 13\delta_3\delta_2 + 11\delta_4\delta_2 + 7\delta_3^2 + 3\delta_4^2 + 13\delta_3\delta_4) \\ & + \frac{9}{2}A^5(3\delta_2 + 4\delta_3 + 3\delta_4) + \frac{9}{2}A^6, \end{aligned} \tag{68}$$

etc, where $\delta_j \equiv x_j - x_{j-1}$. In general, $\phi^{(M)}(x_1, \cdots, x_M)$ is a homogeneous polynomial of $x_1, ..., x_M$ and A of degree $M(M-1)/2$ at $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_M$. One can show that

$$\begin{aligned}\phi^{(M)}(x_1, \cdots, x_M) = {} & |\prod_{i<j}(x_j - x_i)| \\ & + O(A^1 x^{M(M-1)/2-1}) + O(A^2 x^{M(M-1)/2-2}) + \cdots + O(A^{M(M-1)/2-1}x^1) \\ & + 1!\,2!\,\cdots(M-1)!\,M!(A/2)^{M(M-1)/2}. \end{aligned} \tag{69}$$

In particular, we observe

$$\phi^{(M)}(x, \cdots, x) = 1!\,2!\,\cdots(M-1)!\,M!(A/2)^{M(M-1)/2}. \tag{70}$$

5. M-body short-distance correlation of the ideal Fermi gas

Consider a spin-polarized 1D ideal Fermi gas with number density n and temperature T. It has momentum distribution

$$f(k) = \frac{1}{1 + \exp\left(\dfrac{\hbar^2k^2/2m - \mu}{k_B T}\right)}, \tag{71}$$

where m is the mass of each fermion, k_B is Boltzmann's constant, μ is the chemical potential and as before $n = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k)$. Using Wick's theorem, we find that

$$\begin{aligned}\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle = {} & \sum_P (-1)^P \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \cdots \frac{dk_M}{2\pi} f(k_1) \cdots f(k_M) \\ & \times e^{-(ik_1x_1 + \cdots + ik_Mx_M) + (ik_1y_{P1} + \cdots + ik_My_{PM})}, \end{aligned} \tag{72}$$

where $\Psi^\dagger(x)$ and $\Psi(x)$ are respectively the fermion creation and annihilation operators. When the separations between these $2M$ coordinates are much smaller than both the average inter-particle spacing and the thermal de Broglie wave length, we find

$$\begin{aligned}\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle = {} & \frac{I_M}{[1!\,2!\,\cdots(M-1)!]^2 M!} \prod_{i<j}(x_j - x_i)(y_j - y_i) \\ & + (\text{higher-order terms in the separations}), \end{aligned} \tag{73}$$

where

$$I_M \equiv \int \frac{dk_1}{2\pi} \cdots \frac{dk_M}{2\pi} f(k_1) \cdots f(k_M) [\prod_{i<j}(k_j - k_i)]^2. \tag{74}$$

At zero temperature we may use equation (34) and the related formulas in section 3 to deduce

$$I_M = \frac{(\prod_{j=1}^{M-1} j!)^2\, M!}{[\prod_{j=1}^{M-1}(2j-1)!!]^2 (2M-1)!!} \pi^{M(M-1)} n^{M^2}. \tag{75}$$

Let $\hat{n}(x) = \Psi^\dagger(x)\Psi(x)$ be the local number density operator. We find that

$$\langle \hat{n}(x_1) \cdots \hat{n}(x_M)\rangle = \frac{I_M}{[1!\,2!\,\cdots(M-1)!]^2 M!} \prod_{i<j}(x_j - x_i)^2 \tag{76}$$

plus higher-order terms in the separations, at small but non-zero separations.

6. M-body short-distance non-local correlation of the strongly repulsive Bose gas

In this section we concentrate on the non-local M-body correlation function of the 1D strongly repulsive Bose gas, with $\gamma = \frac{c}{n} = \frac{2}{nA} \gg 1$ and in the temperature regime $T \ll \gamma^2 T_d$, where $T_d \equiv \hbar^2 n^2/(2mk_B)$ is the quantum degeneracy temperature, and n is the number density. In such a regime, when $x_1, \cdots, x_M$ are sufficiently close to each other, such that their maximum separation is comparable to or less than A, but the remaining $(N - M)$ particles are not that close to them, the N-body wave function is approximately factorized as

$$\psi(x_1, \cdots, x_M, x_{M+1}, \cdots, x_N) \approx \phi^{(M)}(x_1, \cdots, x_M)\Phi(\bar{x}, x_{M+1}, \cdots, x_N) \tag{77}$$

plus higher-order corrections, where $\bar{x} = (x_1 + \cdots + x_M)/M$.

Assuming that

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_N |\psi(x_1, \cdots, x_N)|^2 = 1, \tag{78}$$

we have

$$\begin{aligned}\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle = \frac{N!}{(N-M)!}\int \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N \\ \times \psi^*(x_1, \cdots, x_M, x_{M+1}, \cdots, x_N) \\ \times \psi(y_1, \cdots, y_M, x_{M+1}, \cdots, x_N),\end{aligned} \tag{79}$$

where $\Psi^\dagger(x)$ and $\Psi(x)$ are respectively the boson creation and annihilation operators. When the maximum separation of $x_1, \cdots, x_M, y_1, \cdots, y_M$ is comparable to or less than A, we may substitute equation (77) into equation (79) to obtain

$$\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\psi(y_M) \cdots \Psi(y_1)\rangle \approx B_M \phi^{(M)*}(x_1, \cdots, x_M)\phi^{(M)}(y_1, \cdots, y_M), \tag{80}$$

where

$$B_M = \frac{N!}{(N-M)!}\int \mathrm{d}x_{M+1} \cdots \mathrm{d}x_N \ |\Phi(\bar{x}, x_{M+1}, \cdots, x_N)|^2. \tag{81}$$

We have ignored the tiny difference between $\bar{x}$ and $\bar{y}$ in equation (80). If the total momentum of the system is zero, B_M is independent of $\bar{x}$.

A special case of equation (80) is

$$\langle \hat{n}(x_1) \cdots \hat{n}(x_M)\rangle \approx B_M \ |\phi^{(M)}(x_1, \cdots, x_M)|^2, \tag{82}$$

if the coordinates $x_1, \cdots, x_M$ do not coincide. Here $\hat{n}(x)$ is the local number density operator of the bosons.

When the separations between $x_1, \cdots, x_M$ are much larger than A, but much smaller than both the average inter-particle spacing and the thermal de Broglie wave length, we can use our knowledge of $\phi^{(M)}$ to deduce that

$$\langle \hat{n}(x_1) \cdots \hat{n}(x_M)\rangle \approx B_M \prod_{i<j} (x_j - x_i)^2.$$

Comparing the above formula with our result for the ideal Fermi gas (see equation (76)), we get

$$B_M = \frac{I_M}{[1!\,2! \cdots (M-1)!]^2 M!}. \tag{83}$$

Therefore, when the separations between $x_1, \cdots, x_M, y_1, \cdots, y_M$ are much smaller than both the average inter-particle spacing and the thermal de Broglie wave length, we get

$$\langle \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_M)\Psi(y_M) \cdots \Psi(y_1)\rangle \approx \frac{I_M}{[1!\,2! \cdots (M-1)!]^2 M!}\phi^{(M)*}(x_1, \cdots, x_M)\phi^{(M)}(y_1, \cdots, y_M), \tag{84}$$

where $\phi^{(M)}$ is defined in section 4. At zero temperature, I_M is given by equation (75). At non-zero temperatures $T \ll \gamma^2 T_d$, one can use equation (74) to calculate I_M. When $T \gtrsim \gamma^2 T_d$, equation (84) breaks down. We emphasize that equation (84) is a *key result* of this paper.

When the above $2M$ coordinates are all equal, we get

$$\langle [\Psi^\dagger(x)]^M [\Psi(x)]^M\rangle \approx M!\, I_M (A/2)^{M(M-1)}. \tag{85}$$

At zero temperature, using equation (75) we find

$$\frac{\langle [\Psi^\dagger(x)]^M [\Psi(x)]^M\rangle}{n^M} = \frac{[\prod_{j=1}^{M} (j!)]^2}{[\prod_{j=1}^{M-1} (2j-1)!!]^2 (2M-1)!!}\left(\frac{\pi}{\gamma}\right)^{M(M-1)} + o(\gamma^{-M(M-1)}) \quad \text{at } T = 0, \tag{86}$$

which is consistent with equation (48).

7. Conclusions

Higher-order quantum correlations reveal the quantum many-body effects in ultracold atomic gases [26]. In light of Gangardt and Shlyapnikov's method for calculating the higher-order correlation functions of 1D bosons with an infinitely strong interaction, we have rigorously calculated the M-body correlation function. It turns out that the quasimomentum distribution correction $\alpha = 1 - 2/\gamma$ to the free fermions leads to the sub-leading terms in the M-body correlation functions at a large interaction strength. We have calculated the higher-order local correlation functions in terms of the statistical parameter α and obtained g_M explicitly for arbitrary M with sub-leading order terms These results not only recover the expressions for g_2 and g_3 with the sub-leading terms given in the literature [1, 2, 4, 51, 52] but also provide explicit forms of g_M with arbitrary M at zero and non-zero temperatures. To our best knowledge, there is not yet another such analytical expression of the local correlation functions g_M (51) for $M > 3$ in the literature for the 1D Bose gas[9]. Moreover, we have explicitly calculated the short-distance non-local M-body correlation functions of the 1D free fermions and the 1D strongly interacting bosons in equations (73) and (84). Our results provide new insights into the many-body correlations in quantum systems of interacting bosons and non-interacting fermions.

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Appendix

In the strong coupling limit, $\gamma \gg 1$, the higher-order correlation function is given by equation (4),

$$g_M = \frac{N!}{(N-M)!}\frac{\int|\psi(0,\cdots,0,\,x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots\,\mathrm{d}x_N}{\int|\psi(x_1,\cdots,x_N)|^2 \mathrm{d}x_1\cdots\,\mathrm{d}x_N}.$$

Calculating the numerator of the formula

The Bethe ansatz energy eigenfunction in the domain $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant L$ is

$$\begin{aligned}\psi(x_1,\,x_2,\cdots,x_N) &= \sum_p (-1)^p \left[\prod_{1\leqslant i<j\leqslant N}\left(1+\frac{\mathrm{i}k_{pj}-\mathrm{i}k_{pi}}{c}\right)\right]\exp\left(\sum_{j=1}^{N}\mathrm{i}k_{pj}x_j\right)\\ &= \left[\prod_{1\leqslant i<j\leqslant N}\left(1+\frac{\partial_{x_j}-\partial_{x_i}}{c}\right)\right]\phi^{(0)}(x_1,\cdots,x_N),\end{aligned} \tag{87}$$

where

$$\phi^{(0)}(x_1,\cdots,x_N) \equiv \sum_p (-1)^p \exp\left(\sum_{j=1}^{N}\mathrm{i}k_{pj}x_j\right), \quad \text{for all } x \tag{88}$$

is completely antisymmetric under the exchange of its arguments. The function

$$\prod_{1\leqslant i\leqslant M;M+1\leqslant j\leqslant N}\left(1+\frac{\partial_{x_j}-\partial_{x_i}}{c}\right)\prod_{M+1\leqslant i<j\leqslant N}\left(1+\frac{\partial_{x_j}-\partial_{x_i}}{c}\right)\phi^{(0)}(x_1,\cdots,x_N) \tag{89}$$

is still antisymmetric under the exchange of any two coordinates x_i and x_j satisfying $1 \leqslant i < j \leqslant M$. Being smooth, such a function must vanish like $\delta^{M(M-1)/2}$ (or even faster) when the coordinates $x_1,\cdots,x_M$ are of the order δ and δ goes to zero. Therefore, in the domain $0 \leqslant x_{M+1} \leqslant x_{M+2} \leqslant \cdots \leqslant x_N \leqslant L$ we have

[9] The integral form of M-body local correlation functions, equation (28) with α set to 1, was presented in [2, 43].

$$\psi(0,\cdots,0,\,x_{M+1},\cdots,x_N) = \prod_{1\leqslant i<j\leqslant M}\frac{\partial_{x_j}-\partial_{x_i}}{c}\prod_{1\leqslant i\leqslant M;M+1\leqslant j\leqslant N}\left(1+\frac{\partial_{x_j}-\partial_{x_i}}{c}\right)$$
$$\prod_{M+1\leqslant i<j\leqslant N}\left(1+\frac{\partial_{x_j}-\partial_{x_i}}{c}\right)\phi^{(0)}(x_1,\cdots,x_N)|_{x_1=\cdots=x_M=0}. \quad (90)$$

At strong coupling, $c\to\infty$, we expand equation (90) as

$$\psi(0,\cdots,0,\,x_{M+1},\cdots,x_N) = \chi^{(0)}(x_{M+1},\cdots,x_N)+\chi^{(1)}(x_{M+1},\cdots,x_N)+O(c^{-M(M-1)/2-2}). \quad (91)$$

In the domain $0\leqslant x_{M+1}\leqslant x_{M+2}\leqslant\cdots\leqslant x_N\leqslant L$ we have

$$\chi^{(0)}(x_{M+1},\cdots,x_N) = c^{-M(M-1)/2}[\prod_{1\leqslant i<j\leqslant M}(\partial_{x_j}-\partial_{x_i})]\phi^{(0)}(x_1,\cdots,x_N)|_{x_1=\cdots=x_M=0},$$

$$\text{and}\quad \chi^{(1)}(x_{M+1},\cdots,x_N) = c^{-M(M-1)/2-1}\left[-(N-M)\sum_{l=1}^{M}\partial_{x_l}+\sum_{l=M+1}^{N}(2l-N-1)\partial_{x_l}\right], \quad (92)$$

$$[\prod_{1\leqslant i<j\leqslant M}(\partial_{x_j}-\partial_{x_i})]\phi^{(0)}(x_1,\cdots,x_N)|_{x_1=\cdots=x_M=0}. \quad (93)$$

Let

$$\phi^{(\Delta)}(\epsilon,\,x_{M+1},\cdots,x_N)\equiv[\prod_{1\leqslant i<j\leqslant M}(\partial_{x_j}-\partial_{x_i})]\phi^{(0)}(x_1,\cdots,x_N)|_{x_1=\cdots=x_M=\epsilon}. \quad (94)$$

Then in the domain $0\leqslant x_{M+1}\leqslant x_{M+2}\leqslant\cdots\leqslant x_N\leqslant L$ equations (92) and (93) become

$$\chi^{(0)}(x_{M+1},\cdots,x_N) = c^{-M(M-1)/2}\phi^{(\Delta)}(0,\,x_{M+1},\cdots,x_N), \quad (95)$$

$$\text{and}\quad \chi^{(1)}(x_{M+1},\cdots,x_N) = c^{-M(M-1)/2-1}\left[-(N-M)\frac{\partial\phi^{(\Delta)}(\epsilon,\,x_{M+1},\cdots,x_N)}{\partial\epsilon}\Big|_{\epsilon=0}\right.$$
$$\left.+\sum_{l=M+1}^{N}(2l-N-1)\frac{\partial\phi^{(\Delta)}(0,\,x_{M+1},\cdots,x_N)}{\partial x_l}\right]. \quad (96)$$

Therefore, in the domain $0\leqslant x_{M+1}\leqslant x_{M+2}\leqslant\cdots\leqslant x_N\leqslant L$ we have

$$\begin{aligned}|\psi(0,\cdots,0,\,x_{M+1},\cdots,x_N)|^2 &= |\chi^{(0)}|^2+\chi^{(0)*}\chi^{(1)}+\chi^{(0)}\chi^{(1)*}+O(c^{-M(M-1)-2})\\ &= c^{-M(M-1)}|\phi^{(\Delta)}(0,\,x_{M+1},\cdots,x_N)|^2\\ &\quad+c^{-M(M-1)-1}\left[-(N-M)\frac{\partial|\phi^{(\Delta)}(\epsilon,\,x_{M+1},\cdots,x_N)|^2}{\partial\epsilon}\Big|_{\epsilon=0}\right.\\ &\quad\left.+\sum_{l=M+1}^{N}(2l-N-1)\frac{\partial|\phi^{(\Delta)}(0,\,x_{M+1},\cdots,x_N)|^2}{\partial x_l}\right]+O(c^{-M(M-1)-2}). \quad (97)\end{aligned}$$

Let

$$\rho^{(\Delta)}(\epsilon)\equiv\int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_N\leqslant L}|\phi^{(\Delta)}(\epsilon,\,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_N, \quad (98)$$

$$\rho_l^{(\Delta)}(x_l)\equiv\int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_N\leqslant L}|\phi^{(\Delta)}(0,\,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_{l-1}\mathrm{d}x_{l+1}\cdots\mathrm{d}x_N, \quad (99)$$

where $M+1\leqslant l\leqslant N$. Then

$$\begin{aligned}&\int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_N\leqslant L}|\psi(0,\cdots,0,\,x_{M+1},\cdots,x_N)|^2\mathrm{d}x_{M+1}\cdots\mathrm{d}x_N\\ &= c^{-M(M-1)}\rho^{(\Delta)}(0)+c^{-M(M-1)-1}\left\{-(N-M)\frac{\partial\rho^{(\Delta)}(\epsilon)}{\partial\epsilon}\,|_{\epsilon=0}\right.\\ &\quad\left.+\sum_{l=M+1}^{N}(2l-N-1)[\rho_l^{(\Delta)}(L)-\rho_l^{(\Delta)}(0)]\right\}\\ &\quad+O(c^{-M(M-1)-2}). \quad (100)\end{aligned}$$

At leading order in $1/c$, the quasimomenta k_j $(1\leqslant j\leqslant N)$ may be approximated as $(2\pi/L)\times$ integers. This implies that $\rho^{(\Delta)}(\epsilon)$ is independent of ϵ at leading order in $1/c$, and

$$\frac{\partial\rho^{(\Delta)}(\epsilon)}{\partial\epsilon}\,|_{\epsilon=0}=O(1/c). \quad (101)$$

Consideration of the volumes of the domain of integration indicate that

$$\rho_l^{(\Delta)}(0) = 0, \quad \text{if } M+2 \leqslant l \leqslant N, \tag{102}$$

$$\rho_l^{(\Delta)}(L) = 0, \quad \text{if } M+1 \leqslant l \leqslant N-1. \tag{103}$$

Because of the smoothness and the complete antisymmetry of $\phi^{(0)}(x_1,\cdots,x_N)$, it is easy to see that when $x_{M+1} \to 0$, the function $\phi^{(\Delta)}(0,\, x_{M+1},\cdots,x_N)$ vanishes like $x_{M+1}^M \to 0$. Consequently

$$\rho_{M+1}^{(\Delta)}(0) = 0. \tag{104}$$

Finally

$$\begin{aligned}\rho_N^{(\Delta)}(L) &= \int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_{N-1}\leqslant L} |\phi^{(\Delta)}(0,\, x_{M+1},\cdots,x_{N-1},\, L)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_{N-1}\\ &= \int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_{N-1}\leqslant L} |\phi^{(\Delta)}(0,\, x_{M+1},\cdots,x_{N-1},\, 0) + O(1/c)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_{N-1}\\ &= \int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_{N-1}\leqslant L} |O(1/c)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_{N-1}\\ &= O(1/c^2).\end{aligned} \tag{105}$$

Combining the above findings, we simplify equation (100) as

$$\begin{aligned}&\int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_N\leqslant L} |\psi(0,\cdots,0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N\\ &= c^{-M(M-1)}\rho^{(\Delta)}(0) + O(c^{-M(M-1)-2})\\ &= c^{-M(M-1)}\int_{0\leqslant x_{M+1}\leqslant\cdots\leqslant x_N\leqslant L} |\phi^{(\Delta)}(0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N + O(c^{-M(M-1)-2}).\end{aligned} \tag{106}$$

Multiplying the above equation by $(N-M)!$, we get

$$\begin{aligned}&\int_0^L |\psi(0,\cdots,0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N\\ &= c^{-M(M-1)}\int_0^L |\phi^{(\Delta)}(0,\, x_{M+1},\cdots,x_N)|^2 \mathrm{d}x_{M+1}\cdots \mathrm{d}x_N + O(c^{-M(M-1)-2}).\end{aligned} \tag{107}$$

Calculating the denominator of the formula

At strong coupling, $c \to \infty$, we expand equation (87) as

$$\psi(x_1,\cdots,x_N) = \psi^{(0)}(x_1,\cdots,x_N) + \psi^{(1)}(x_1,\cdots,x_N) + O(c^{-2}), \tag{108}$$

where in the domain $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant L$ we have

$$\psi^{(0)}(x_1,\cdots,x_N) = \sum_p (-1)^p \exp\left(\sum_{j=1}^N \mathrm{i}k_{pj}x_j\right), \tag{109}$$

$$\begin{aligned}\text{and}\quad \psi^{(1)}(x_1,\cdots,x_N) &= \sum_{1\leqslant i<j\leqslant N} \frac{(\partial_{x_j} - \partial_{x_i})}{c}\psi^{(0)}(x_1,\cdots,x_N)\\ &= \frac{1}{c}\sum_{j=1}^N (2j-N-1)\partial_{x_j}\psi^{(0)}(x_1,\cdots,x_N).\end{aligned} \tag{110}$$

Therefore

$$\begin{aligned}\int_0^L |\psi(x_1,\cdots,x_N)|^2 \mathrm{d}x_1\cdots \mathrm{d}x_N &= N!\int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} |\psi^{(0)} + \psi^{(1)}|^2 \mathrm{d}x_1\cdots \mathrm{d}x_N + O(c^{-2})\\ &= N!\int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} [|\psi^{(0)}|^2 + (\psi^{(0)*}\psi^{(1)} + \psi^{(0)}\psi^{(1)*})]\mathrm{d}x_1\cdots \mathrm{d}x_N + O(c^{-2})\\ &= N!\int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} |\psi^{(0)}|^2 \mathrm{d}x_1\cdots \mathrm{d}x_N\\ &\quad + \frac{2(N!)}{c}\sum_{j=1}^N (2j-N-1)\mathrm{Re}\, b_j + O(c^{-2}),\end{aligned} \tag{111}$$

where

$$b_j \equiv \int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} \psi^{(0)*}\partial_{x_j}\psi^{(0)}\mathrm{d}x_1\cdots\mathrm{d}x_N. \tag{112}$$

Since the total momentum $(k_1+\cdots+k_N)$ must be an integer times $(2\pi/L)$, it is easy to see that when $0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L$, we have

$$\begin{aligned}\psi^{(0)}(L-x_N,\,L-x_{N-1},\cdots,L-x_1) &= \sum_p(-1)^p\exp\left[\mathrm{i}\sum_{j=1}^N k_{pj}(L-x_{N-j+1})\right]\\ &= \sum_p(-1)^p\exp\left(-\mathrm{i}\sum_{j=1}^N k_{pj}x_{N-j+1}\right)\\ &= (-1)^r\psi^{(0)*}(x_1,\,x_2,\cdots,x_N),\end{aligned} \tag{113}$$

where $(-1)^r$ is the signature of the reversal permutation $\{1,\cdots,N\}\to\{N,\cdots,1\}$. In particular, $(-1)^r=+1$ if $\mathrm{mod}(N,\,4)=0$ or 1, and $(-1)^r=-1$ if $\mathrm{mod}(N,\,4)=2$ or 3.

From the above equation one can show that

$$b_{N+1-j} = -b_j^*. \tag{114}$$

Define

$$\rho_j(x) = \int_{0\leqslant x_1\leqslant\cdots\leqslant x_{j-1}\leqslant x\leqslant x_{j+1}\leqslant\cdots\leqslant x_N\leqslant L} |\psi^{(0)}(x_1,\cdots,x_{j-1},\,x,\,x_{j+1},\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_{j-1}\mathrm{d}x_{j+1}\cdots\mathrm{d}x_N. \tag{115}$$

It is easy to see that

$$2\,\mathrm{Re}\,b_j = \int_0^L \frac{\partial\rho_j(x)}{\partial x}\mathrm{d}x = \rho_j(L)-\rho_j(0). \tag{116}$$

From the definition of $\rho_j(x)$ we can easily see that

$$\rho_j(0) = 0, \quad \text{if } j\geqslant 2, \tag{117}$$

$$\text{and}\quad \rho_j(L) = 0, \quad \text{if } j\leqslant N-1. \tag{118}$$

It is also easy to see that

$$\rho_1(0) = \rho_N(L). \tag{119}$$

Thus

$$2\,\mathrm{Re}\,b_j = 0,\ \text{if } 2\leqslant j\leqslant N-1, \tag{120}$$

and, assuming that $N\geqslant 2$, we have

$$2\,\mathrm{Re}\,b_1 = -\rho_N(L), \tag{121}$$

$$2\,\mathrm{Re}\,b_N = +\rho_N(L). \tag{122}$$

So, equation (111) is simplified as

$$\int_0^L |\psi(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N = \int_0^L |\psi^{(0)}|^2\mathrm{d}x_1\cdots\mathrm{d}x_N + \frac{N!\,2(N-1)}{c}\rho_N(L)+O(c^{-2}). \tag{123}$$

Let

$$\rho_{\mathrm{arb}}(x) \equiv \int_0^L |\psi^{(0)}(x_1,\cdots,x_{N-1},\,x)|^2\mathrm{d}x_1\cdots\mathrm{d}x_{N-1}. \tag{124}$$

Strictly speaking, $\rho_{\mathrm{arb}}(x)$ depends on x. But in the large c limit if we approximate $\psi^{(0)}$ by ψ, then $\rho_{\mathrm{arb}}(x)$ is approximately proportional to the local number density, which is a constant at thermal equilibrium. Thus

$$\rho_{\mathrm{arb}}(x) = [1+O(1/c)]\frac{1}{L}\int_0^L \rho_{\mathrm{arb}}(y)\mathrm{d}y = [1+O(1/c)]\frac{1}{L}\int_0^L |\psi^{(0)}|^2\mathrm{d}x_1\cdots\mathrm{d}x_N, \quad \text{if } 0\leqslant x\leqslant L. \tag{125}$$

On the other hand, it is easy to see that

$$\rho_N(L) = \frac{1}{(N-1)!}\rho_{\mathrm{arb}}(L). \tag{126}$$

Thus, in the limit $c\to\infty$ we have

$$\int_0^L |\psi(x_1,\cdots,x_N)|^2\mathrm{d}x_1\cdots\mathrm{d}x_N = \left[1+\frac{2N(N-1)}{cL}+O(c^{-2})\right]\int_0^L |\psi^{(0)}|^2\mathrm{d}x_1\cdots\mathrm{d}x_N. \tag{127}$$

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