University of Ruhuna - Faculty of Science
Bachelor of Science General Degree
Level I (Semester I) Examination - June 2023

Subject: Mathematics
Course Unit: MAT111β (Vector Analysis) **Time:** Two (02) Hours

Answer **ALL Questions**

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1. Consider the two planes π_1 and π_2 given by

$$\pi_1 : x - 3y + 6z = 4,$$
$$\pi_2 : 5x + y - z = 4$$

in the three dimensional space.

Write down the normal vectors $\underline{n}_1$ and $\underline{n}_2$ for the planes π_1 and π_2 respectively. [10]

(a) Determine whether the planes π_1 and π_2

(i) are orthogonal or, [15]

(ii) are parallel or, [15]

(iii) intersect. [05]

(b) If π_1 and π_2 intersect, find the

(i) angle of intersection and [15]

(ii) set of parametric equations for the line of intersection of the planes. [25]

(c) The plane π_2 and the line

$$x = 1 + t,$$
$$y = -2 + 2t,$$
$$z = 4t$$

intersect at the point P with coordinates (x_0, y_0, z_0). Find the coordinates of P. [15]

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2. (a) Suppose that $f(x,y) = \tan(xy)$ is a two variable scalar function. Find a unit vector, $\hat{u}$ such that the directional derivative of f in the direction of $\hat{u}$ at the point $(1,\pi)$ is zero, i.e. $D_{\hat{u}} f(1,\pi) = 0$. [25]

(b) State, in the usual notation, the Green's theorem in the plane. [10]

Show that

$$\oint_C \frac{y}{x+1}\,dx + 2xy\,dy = 2\ln 2 - \frac{41}{30},$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. [35]

Using the above result, verify the Green's theorem. [30]

3. State the divergence theorem of Gauss, explaining clearly the symbols and notations you use.

   Let $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$ be a vector field and S be the surface of the **cuboid (solid rectangle)** defined by

$$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$$

   Verify the divergence theorem of Gauss for this case.

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4. (a) Consider the vector field $\underline{F} = 2xy^3\underline{i} + 3x^2y^2\underline{j} + (x+z)\underline{k}$ and the curve C consisting of the line segments joining $A\,(2,0,0)$ to $B\,(0,1,0)$ to $D\,(0,0,1)$ back to A.

   (i) Calculate the curl of $\underline{F}$. [15]

   (ii) Find a unit normal vector $\hat{n}$ for the triangular plane region ABD. [15]

   (iii) Find the area S of the triangle. [15]

   (iv) Apply the Stoke's theorem and the results in (i), (ii) and (iii) to evaluate the line integral

$$\oint_C \underline{F}\cdot d\underline{r}.$$

   [15]

   (b) Let $\underline{F} = F_u\underline{e}_u + F_v\underline{e}_v + F_w\underline{e}_w$ be a vector field. Write down, in the usual notation, the expressions for $\operatorname{div}\underline{F}$ and $\operatorname{curl}\underline{F}$ in **orthogonal curvilinear coordinates** (u,v,w). Here, $\underline{e}_u, \underline{e}_v$ and $\underline{e}_w$ have usual meanings. [10]

   Consider the vector field

$$\underline{F} = r\sin\theta\,\underline{e}_r + r\cos\phi\,\underline{e}_\theta + r\sin\phi\,\underline{e}_\phi$$

   given in terms of spherical polar coordinates (r,θ,ϕ).

   (i) Find $\operatorname{div}\underline{F}$. [15]

   (ii) Show also that

$$\operatorname{curl}\underline{F} = \sin\phi\,\cot(\theta/2)\,\underline{e}_r - 2\sin\phi\,\underline{e}_\theta + (2\cos\phi - \cos\theta)\,\underline{e}_\phi.$$

   [15]
