UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: March 2026

Module Number: IS3301 **Module Name: Complex Analysis and Mathematical Transforms (C 23)**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

Q1. a) Discuss the continuity of the function $f(z)$ at $z = 0$;

$$f(z) = \begin{cases} \dfrac{|z|^2}{z} & ; \ z \neq 0 \\ 0 & ; \ z = 0 \end{cases}$$

[3 Marks]

b) Let the function $f(z) = u(x, y) + iv(x, y)$ be an analytic function with the harmonic function $u(x, y) = x^2 - y^2 + 4y$.

i Find a harmonic conjugate of $u(x, y)$.

ii Find the analytic function $f(z)$.

[4 Marks]

c) i State any two elementary conformal transformations.

ii Using a linear fractional transformation, find the image of the upper half-plane $\mathrm{Im}(z) > 0$ under $w = \frac{z-i}{z+i}$ considering separately the cases $z = x$, and $z = x + iy$, where $x \in \mathbb{R}$, and $y > 0$.

[5 Marks]

Q2. a) Find the Taylor series expansion of $g(z) = e^z$ about the point $z = i$.

[3 Marks]

b) Identify all singular points of each function that lie inside the circle $|z| = 3$.

i $f_1(z) = \dfrac{1}{(z-2)^3}$

ii $f_2(z) = \dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$

iii $f_3(z) = \dfrac{z}{(z-3)(z^2+4)^2}$

[4 Marks]

c) State the Cauchy's Residue Theorem in standard notation, and hence, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2+1)(z^2+4)}\, dz$$

by considering the following contours:

i A semicircular contour lying entirely above the real axis in the complex plane.

ii A semicircular contour lying entirely to the left of the imaginary axis in the complex plane. Justify your answer.

[5 Marks]

Q3. a) State the definition of the Laplace Transform of a function $f(t)$. Hence, find the Laplace Transform of the following functions:

i $f(t) = 1$

ii $f(t) = e^{at}$, where a is a constant

[3 Marks]

b) Using the Laplace Transform, solve the following initial value problem:

$$y'' + 3y' + 2y = e^{-t}, y(0) = 0, y'(0) = 1$$

[4 Marks]

c) State the Convolution Theorem for Laplace Transforms.
Let $f(t) = t$ and $g(t) = e^{2t}$ for $t \geq 0$. Using the convolution theorem, find the Laplace Transform of the function $h(t) = (f * g)(t)$.

[5 Marks]

Q4. a) Define the Z-Transform of a discrete-time sequence $x[n]$. Hence, find the Z-Transform of the following basic sequences:

i $x[n] = 1, n \geq 0$

ii $x[n] = a^n, n \geq 0, a$ is a constant

[3 Marks]

b) Solve the following first-order difference equation using the Z-transform method

$$y[n] - 3y[n-1] = 4, n \geq 0$$

with the initial condition, $y[-1] = 1$.

[4 Marks]

c) State the Convolution Theorem for Z-Transforms.
Let $x[n] = 1$ and $h[n] = a^n$ for $n \geq 0$, where $a$is a constant. Using the Z-Transform and the convolution theorem, find the z transform of $y[n] = (x * h)[n]$.

[5 Marks]

Q5. a) Consider the Fourier Series for a function $f(t)$ of period 2π;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt);$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt$; $n = 0,1,2,3, \ldots,$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt ; n = 1,2,3, \ldots$

Let $f(t) = \pi - t, 0 < t < \pi$.

i Using the half-range cosine series formulas, compute the Fourier coefficients and obtain the half-range cosine series for $f(t)$ assuming period 2π.

ii By choosing a suitable value of t, show that:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

[5 Marks]

b) i Define the term 'absolutely integrable function' for a function defined on $(-\infty, \infty)$.

ii Find the Fourier transform of the unit step function $u(t)$, where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Give a clear justification for your answer.
[Hint: $\lim_{\epsilon \to 0^+} \frac{\epsilon}{\epsilon^2 + \omega^2} = \pi\delta(\omega)$]

iii Using the result from part (ii), evaluate the Fourier transform of $f(t) = t\,u(t)$
[Hint: $\mathcal{F}\{tf(t)\} = i\frac{d}{d\omega}\mathcal{F}\{f(t)\}$]

[7 Marks]

Table of Laplace Transform Pairs

	$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{c-jT}^{c+jT} F(s)e^{st}ds$		$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$	
transform	$f(t)$	$\xleftrightarrow{\mathcal{L}}$	$F(s)$	
complex conjugation	$f^*(t)$	$\xleftrightarrow{\mathcal{L}}$	$F^*(s^*)$	
time shifting	$f(t-a) \quad t \geqslant a > 0$	$\xleftrightarrow{\mathcal{L}}$	$a^{-as}F(s)$	
	$e^{-at}f(t)$	$\xleftrightarrow{\mathcal{L}}$	$F(s+a)$	frequency shifting
time scaling	$f(at)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	
linearity	$af_1(t)+bf_2(t)$	$\xleftrightarrow{\mathcal{L}}$	$aF_1(s)+bF_2(s)$	
time multiplication	$f_1(t)f_2(t)$	$\xleftrightarrow{\mathcal{L}}$	$F_1(s)*F_2(s)$	frequency convolution
time convolution	$f_1(t)*f_2(t)$	$\xleftrightarrow{\mathcal{L}}$	$F_1(s)F_2(s)$	frequency product
delta function	$\delta(t)$	$\xleftrightarrow{\mathcal{L}}$	1	
shifted delta function	$\delta(t-a)$	$\xleftrightarrow{\mathcal{L}}$	e^{-as}	exponential decay
unit step	$u(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{s}$	
ramp	$tu(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{s^2}$	
parabola	$t^2u(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{2}{s^3}$	
n-th power	t^n	$\xleftrightarrow{\mathcal{L}}$	$\frac{n!}{s^{n+1}}$	
exponential decay	e^{-at}	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{s+a}$	
two-sided exponential decay	$e^{-a\lvert t\rvert}$	$\xleftrightarrow{\mathcal{L}}$	$\frac{2a}{a^2-s^2}$	
	te^{-at}	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{(s+a)^2}$	
	$(1-at)e^{-at}$	$\xleftrightarrow{\mathcal{L}}$	$\frac{s}{(s+a)^2}$	
exponential approach	$1-e^{-at}$	$\xleftrightarrow{\mathcal{L}}$	$\frac{a}{s(s+a)}$	
sine	$\sin(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{\omega}{s^2+\omega^2}$	
cosine	$\cos(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{s}{s^2+\omega^2}$	
hyperbolic sine	$\sinh(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{\omega}{s^2-\omega^2}$	
hyperbolic cosine	$\cosh(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{s}{s^2-\omega^2}$	
exponentially decaying sine	$e^{-at}\sin(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{\omega}{(s+a)^2+\omega^2}$	
exponentially decaying cosine	$e^{-at}\cos(\omega t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{s+a}{(s+a)^2+\omega^2}$	
frequency differentiation	$tf(t)$	$\xleftrightarrow{\mathcal{L}}$	$-F'(s)$	
frequency n-th differentiation	$t^nf(t)$	$\xleftrightarrow{\mathcal{L}}$	$(-1)^nF^{(n)}(s)$	
time differentiation	$f'(t) = \frac{d}{dt}f(t)$	$\xleftrightarrow{\mathcal{L}}$	$sF(s)-f(0)$	
time 2nd differentiation	$f''(t) = \frac{d^2}{dt^2}f(t)$	$\xleftrightarrow{\mathcal{L}}$	$s^2F(s)-sf(0)-f'(0)$	
time n-th differentiation	$f^{(n)}(t) = \frac{d^n}{dt^n}f(t)$	$\xleftrightarrow{\mathcal{L}}$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$	
time integration	$\int_0^t f(\tau)d\tau = (u*f)(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{1}{s}F(s)$	
frequency integration	$\frac{1}{t}f(t)$	$\xleftrightarrow{\mathcal{L}}$	$\int_s^\infty F(u)du$	
time inverse	$f^{-1}(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{F(s)-f^{-1}}{s}$	
time differentiation	$f^{-n}(t)$	$\xleftrightarrow{\mathcal{L}}$	$\frac{F(s)}{s^n}+\frac{f^{-1}(0)}{s^n}+\frac{f^{-2}(0)}{s^{n-1}}+\ldots+\frac{f^{-n}(0)}{s}$	

Source: Marc Ph. Stoecklin, TABLES OF TRANSFORM PAIRS, v1.5.3

Table of z-Transform Pairs

	$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$		$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	ROC
transform	$x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(z)$	R_x
time reversal	$x[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(\frac{1}{z})$	$\frac{1}{R_x}$
complex conjugation	$x^*[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(z^*)$	R_x
reversed conjugation	$x^*[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(\frac{1}{z^*})$	$\frac{1}{R_x}$
real part	$\Re\mathfrak{e}\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	R_x
imaginary part	$\Im\mathfrak{m}\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	R_x
time shifting	$x[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$z^{-n_0}X(z)$	R_x
scaling in $\mathcal{Z}$	$a^n x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X\left(\frac{z}{a}\right)$	$\|a\|R_x$
downsampling by N	$x[Nn], N \in \mathbb{N}_0$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{N}\sum_{k=0}^{N-1} X\left(W_N^k z^{\frac{1}{N}}\right) \quad W_N = e^{-\frac{j2\omega}{N}}$	R_x
linearity	$ax_1[n] + bx_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$aX_1(z) + bX_2(z)$	$R_x \cap R_y$
time multiplication	$x_1[n]x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2\pi j}\oint X_1(u)X_2\left(\frac{z}{u}\right)u^{-1}du$	$R_x \cap R_y$
frequency convolution	$x_1[n] * x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X_1(z)X_2(t)$	$R_x \cap R_y$
delta function	$\delta[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	1	$\forall z$
shifted delta function	$\delta[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	z^{-n_0}	$\forall z$
step	$u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| > 1$
	$-u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| < 1$
ramp	$nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
	$n^2u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
	$-n^2u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| < 1$
	$n^3u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
	$-n^3u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| < 1$
	$(-1)^n$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z+1}$	$\|z\| < 1$
exponential	$a^nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
	$-a^nu[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
	$a^{n-1}u[n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{z-a}$	$\|z\| > \|a\|$
	$na^nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
	$n^2a^nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az(z+a}{(z-a)^3}$	$\|z\| > \|a\|$
	$e^{-an}u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-e^{-a}}$	$\|z\| > \|e^{-a}\|$
exp. interval	$\begin{cases} a^n & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
sine	$\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z\sin(\omega_0)}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
cosine	$\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
	$a^n\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
	$a^n\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-a\cos(\omega_0))}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
differentiation in $\mathcal{Z}$	$nx[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-z\frac{dX(z)}{dz}$	R_x
integration in $\mathcal{Z}$	$\frac{x[n]}{n}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-\int_0^z \frac{X(z)}{z}dz$	R_x
	$\frac{\prod_{i=1}^{m}(n-i+1)}{a^m m!}a^m u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-a)^{m+1}}$	