# UNIVERSITY OF RUHUNA

## Faculty of Engineering

End-Semester 5 Examination in Engineering: March 2026

**Module Number: IS5302** **Module Name: Numerical Methods(C-23)**

**[Three hours]**

[Answer **all questions,** each question carries twelve marks]

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Q1 In technical analysis, we often encounter functions which involve both linear and trigonometric terms. Consider the function $f(x)$ defined as:

$$f(x) = 2x - \cos x - 1$$

We wish to find the value of $x$ (in radians) where $f(x) = 0$.

a) i.) Use the Intermediate Value Theorem to prove that at least one root exists in the interval $[0.5, 1.0]$.

ii.) Calculate the exact number of iterations required for the Bisection Method to achieve an absolute error tolerance of $10^{-4}$ starting from the interval $[0.5, 1.0]$.

iii.) Perform **four iterations** of the Bisection Method. Present your results in a table showing $a$, $b$, the midpoint $c$, and $f(c)$. Carry all calculations to five decimal places.

[5 Marks]

b) The Bisection Method is reliable but slow. To find a high-precision solution for a computational model, we switch to the Newton-Raphson Method.

i.) Derive the iterative formula of the Newton-Raphson Method for solving the equation

$$f(x) = 2x - \cos x - 1.$$

ii.) Taking the fourth iteration value obtained in Part a) iii.) as the initial approximation $x_0$, apply the Newton-Raphson Method to compute the next **two iterations.**

Give your answers correct to five decimal places.

[3 Marks]

c) i.) The equation $2x - \cos x - 1 = 0$ can be rearranged into the fixed-point form $x = g(x)$ in several ways. Consider:

$$g(x) = \frac{\cos x + 1}{2}.$$

Obtain $g'(x)$ and evaluate its magnitude at $x = 0.8$. Based on the **Fixed-Point Convergence Theorem**, will this iteration converge to the root?

ii.) In professional software, why might a programmer choose the **Bisection Method** even if **Newton-Raphson** is faster?

[4 Marks]

Q2 a) An engineering experiment provides the following data points:

| $i$ | $x_i$ | $f(x_i)$ |
|---|---|---|
| 0 | 2.0 | 0.5 |
| 1 | 2.5 | 0.4 |
| 2 | 4.0 | 0.25 |

i.) Using the Lagrange method, construct the second-degree polynomial $P_2(x)$ to estimate $f(3)$. Round the final estimate to five decimal places.

ii.) If the data follow the function $f(x) = \frac{1}{x}$, determine the true value of $f(3)$ and calculate the absolute error.

iii.) Briefly explain how this error could be reduced in a practical engineering scenario (e.g., changing the data sampling).

[5 Marks]

b) The general Newton Divided Difference polynomial is given by:

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3]$$
$$(x-x_0)(x-x_1)(x-x_2) + \cdots + f[x_0,x_1,x_2,x_3,\dots,x_n](x-x_0)\dots(x-x_{n-1}).$$

i.) For equally spaced points with step size $h$, use the definition of divided differences to show that:

$$f[x_0,x_1,x_2,x_3,\dots,x_n] = \frac{\Delta^k y_0}{k!\,h^k}$$

ii.) Using the substitution $s = \frac{x-x_0}{h}$, show how the general formula in part b) is used to derive the **Newton's Forward Difference Interpolation Formula:**

$$P_n(x) = y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_0}{2!} + s(s-1)(s-2)\frac{\Delta^3 y_0}{3!} + \cdots$$
$$+ s(s-1)(s-2)\dots(s-(n-1))\frac{\Delta^n y_0}{n!}.$$

[3 Marks]

c) An engineering firm is modeling the heat distribution across a metal plate. The steady-state temperatures $(T_1, T_2, T_3)$ at three specific nodes are governed by the following system of linear equations:

$$10T_1 + T_2 + 2T_3 = 44$$

$$2T_1 + 10T_2 + T_3 = 51$$

$$T_1 + 2T_2 + 10T_3 = 61$$

i.) Show that the system is **diagonally dominant.**

ii.) Rearrange the equations into the iterative form required for the Gauss-Seidel method.

iii.) Starting with an initial guess $T^{(0)} = [0,0,0]^T$ perform **two iterations** of the Gauss-Seidel Method. Carry out all calculations to four decimal places.

iv.) Calculate the absolute relative error for $T_1$ after the second iteration using

$$\varepsilon_a{}^{(2)} = \left|\frac{T_2 - T_1}{T_2}\right| \times 100\%.$$

[4 Marks]

Q3 A hydroelectric facility is being evaluated for efficiency. The system performance is captured through three different numerical lenses: reservoir capacity, energy conversion, and mechanical gate timing.

a) The flow rate of water entering the reservoir during a storm is given by

$$Q(t) = 20\ln(t + 3)$$

in $m^3/s$.

i.) Determine the total volume of water ($V$) that entered the reservoir from $t = 0$ to $t = 12$ seconds using the **Trapezoidal Rule** with $\boldsymbol{n} = \mathbf{4}$ **subintervals.** Carry out calculations to four decimal places.

$$V = \int_0^{12} 20\ln(t + 3)\ dt$$

ii.) Calculate the absolute relative error given the exact integral value is 513.5612.

[5 Marks]

b) As water passes through the generator, the power output $P(t)$ in Megawatts (MW) follows a non-linear efficiency curve:

$$P(t) = \frac{15}{2 + t^2}$$

i.) Use the **two-point Gauss quadrature rule** to approximate the total energy produced ($E$) over the first 3 seconds ($E = \int_0^3 P(t)\ dt$). You must transform the limits to $[-1, 1]$.

ii.) Calculate the absolute relative error given the exact value is 14.8210.

Refer to the following table for the weighting factors and function argument values used in the Gauss quadrature rule.

| Point | Weight Factors | Function Arguments |
|---|---|---|
| 2 | $c_1 = 1.0000$ | $x_1 = -0.5773$ |
| | $c_2 = 1.0000$ | $x_2 = 0.5773$ |
| 3 | $c_1 = 0.5555$ | $x_1 = -0.7746$ |
| | $c_2 = 0.8888$ | $x_2 = 0.0000$ |
| | $c_3 = 0.5555$ | $x_3 = 0.7746$ |

[5 Marks]

c) To control the flow, a mechanical sluice gate moves to various positions $s$ (meters) at times $t$ (seconds). The following sensor data were recorded:

| $t\ (s)$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|
| $s\ (m)$ | 0.1987 | 0.2955 | 0.3894 | 0.4794 | 0.5646 |

Approximate the instantaneous velocity of the gate at $t = 0.4s$ using the **central difference formula** with $h = 0.1$.

[2 Marks]

Q4 a) Briefly explain the following by giving an example for each item.

i.) Ordinary differential equations (ODE)

ii.) Initial value problems (IVP)

[2 Marks]

b) A polluted lake has an initial concentration of a bacteria of $10^7$ parts/m$^3$, while the acceptable level is only $10^5$ parts/m$^3$. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration $C$ of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.08Ct = 0, \quad C(0) = 10^7$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

[7 Marks]

c) Using the following ordinary differential equation with the given initial condition and step size $h$, show that the error of 4th order Runge-Kutta method is of order 5 [$O(h^5)$].

$$\frac{dy}{dx} = y, \qquad y(0) = 1$$

[3 Marks]

Q5 a) Classify the following equations as linear or non-linear, and state their order.

i.) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$$

ii.) $$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0$$

iii.) $$2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial x} + \cos 2t = 0$$

iv.) $$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

[2 Marks]

b) i.) List advantages and disadvantages of using the explicit method in solving partial differential equations.

ii.) Use explicit method to solve the partial differential equation,

$$\frac{\partial^2 u(x,t)}{\partial x^2} - 2\frac{\partial u(x,t)}{\partial t} = 0 \text{ ;for } 0 \le x \le 2 \text{ and } 0 \le t \le 0.6$$

with the initial conditions,

$$u(x,0) = f(x) = x\,(2 - x)$$

and the boundary conditions,

$$u(0,t) = 0$$

$$u(2,t) = t^2.$$

Use, $h = 0.5$ and $k = 0.2$, where $h$ and $k$ are step sizes along $x$ and $t$ axes respectively.

[7 Marks]

c) Briefly explain the procedure of finite difference solution technique for solving Laplace equation $\left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0\right)$ highlighting any differences with the solution technique used in part (b).

[3 Marks]