



UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: March 2026

Module Number: IS5306

Module Name: Numerical Methods(C-18)

[Three hours]

[Answer all questions, each question carries twelve marks]

Q1 In technical analysis, we often encounter functions which involve both linear and trigonometric terms. Consider the function $f(x)$ defined as:

$$f(x) = 2x - \cos x - 1$$

We wish to find the value of x (in radians) where $f(x) = 0$.

- a) i.) Use the Intermediate Value Theorem to prove that at least one root exists in the interval $[0.5, 1.0]$.
- ii.) Calculate the exact number of iterations required for the Bisection Method to achieve an absolute error tolerance of 10^{-4} starting from the interval $[0.5, 1.0]$.
- iii.) Perform **four iterations** of the Bisection Method. Present your results in a table showing a , b , the midpoint c , and $f(c)$. Carry all calculations to five decimal places.

[5 Marks]

- b) The Bisection Method is reliable but slow. To find a high-precision solution for a computational model, we switch to the Newton-Raphson Method.
- i.) Derive the iterative formula of the Newton-Raphson Method for solving the equation

$$f(x) = 2x - \cos x - 1.$$

- ii.) Taking the fourth iteration value obtained in Part a) iii.) as the initial approximation x_0 , apply the Newton-Raphson Method to compute the next **two iterations**.

Give your answers correct to five decimal places.

[3 Marks]

- c) i.) The equation $2x - \cos x - 1 = 0$ can be rearranged into the fixed-point form $x = g(x)$ in several ways. Consider:

$$g(x) = \frac{\cos x + 1}{2}.$$

Obtain $g'(x)$ and evaluate its magnitude at $x = 0.8$. Based on the **Fixed-Point Convergence Theorem**, will this iteration converge to the root?

- ii.) In professional software, why might a programmer choose the **Bisection Method** even if **Newton-Raphson** is faster?

[4 Marks]

- Q2 a) An engineering experiment provides the following data points:

i	x_i	$f(x_i)$
0	2.0	0.5
1	2.5	0.4
2	4.0	0.25

- i.) Using the Lagrange method, construct the second-degree polynomial $P_2(x)$ to estimate $f(3)$. Round the final estimate to five decimal places.
- ii.) If the data follow the function $f(x) = \frac{1}{x}$, determine the true value of $f(3)$ and calculate the absolute error.
- iii.) Briefly explain how this error could be reduced in a practical engineering scenario (e.g., changing the data sampling).

[5 Marks]

b) i.) Given

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0) \dots (x - x_{n-1}).$$

Use $P_n(x_2)$ to show that $a_2 = f[x_0, x_1, x_2]$

ii.) If a, b, c, d are the arguments for x of $f(x) = \frac{1}{x^2}$, show that

$$f[a, b, c, d] = -\left(\frac{abc + bcd + acd + adb}{a^2b^2c^2d^2}\right)$$

[3 Marks]

c) An engineering firm is modeling the heat distribution across a metal plate. The steady-state temperatures (T_1, T_2, T_3) at three specific nodes are governed by the following system of linear equations:

$$10T_1 + T_2 + 2T_3 = 44$$

$$2T_1 + 10T_2 + T_3 = 51$$

$$T_1 + 2T_2 + 10T_3 = 61$$

i.) Show that the system is **diagonally dominant**.

ii.) Rearrange the equations into the iterative form required for the Gauss-Seidel method.

iii.) Starting with an initial guess $T^{(0)} = [0, 0, 0]^T$ perform **two iterations** of the Gauss-Seidel Method. Carry out all calculations to four decimal places.

iv.) Calculate the absolute relative error for T_1 after the second iteration using

$$\varepsilon_a^{(2)} = \left| \frac{T_2 - T_1}{T_2} \right| \times 100\%.$$

[4 Marks]

Q3 A hydroelectric facility is being evaluated for efficiency. The system performance is captured through three different numerical lenses: reservoir capacity, energy conversion, and mechanical gate timing.

- a) The flow rate of water entering the reservoir during a storm is given by

$$Q(t) = 20 \ln(t + 3)$$

in m^3/s .

- i.) Determine the total volume of water (V) that entered the reservoir from $t = 0$ to $t = 12$ seconds using the **Trapezoidal Rule** with $n = 4$ subintervals. Carry calculations to four decimal places.

$$V = \int_0^{12} 20 \ln(t + 3) dt$$

- ii.) Calculate the absolute relative error given the exact integral value is 513.5612.

[5 Marks]

- b) As water passes through the generator, the power output $P(t)$ in Megawatts (MW) follows a non-linear efficiency curve:

$$P(t) = \frac{15}{2 + t^2}$$

- i.) Use the **two-point Gauss quadrature rule** to approximate the total energy produced (E) over the first 3 seconds ($E = \int_0^3 P(t) dt$). You must transform the limits to $[-1, 1]$.
- ii.) Calculate the absolute relative error given the exact value is 14.8210.

Refer to the following table for the weighting factors and function argument values used in the Gauss quadrature rule.

Point	Weight Factors	Function Arguments
2	$c_1 = 1.0000$	$x_1 = -0.5773$
	$c_2 = 1.0000$	$x_2 = 0.5773$
3	$c_1 = 0.5555$	$x_1 = -0.7746$
	$c_2 = 0.8888$	$x_2 = 0.0000$
	$c_3 = 0.5555$	$x_3 = 0.7746$

[5 Marks]

- c) To control the flow, a mechanical sluice gate moves to various positions s (meters) at times t (seconds). The following sensor data were recorded:

t (s)	0.2	0.3	0.4	0.5	0.6
s (m)	0.1987	0.2955	0.3894	0.4794	0.5646

Approximate the instantaneous velocity of the gate at $t = 0.4s$ using the **central difference formula** with $h = 0.1$.

[2 Marks]

- Q4 a) Briefly explain the following by giving an example for each item.

- i.) Ordinary differential equations (ODE)
- ii.) Initial value problems (IVP)

[2 Marks]

- b) A polluted lake has an initial concentration of a bacteria of 10^7 parts/m³, while the acceptable level is only 10^5 parts/m³. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.08Ct = 0, \quad C(0) = 10^7$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

[7 Marks]

- c) Using the following ordinary differential equation with the given initial condition and step size h , show that the error of 4th order Runge-Kutta method is of order 5 [$O(h^5)$].

$$\frac{dy}{dx} = y, \quad y(0) = 1$$

[3 Marks]

Q5 a) Classify the following equations as linear or non-linear, and state their order.

i.) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$

ii.) $\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w \frac{\partial w}{\partial x} = 0$

iii.) $2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3 \frac{\partial^2 f}{\partial t^2} + 4 \frac{\partial f}{\partial x} + \cos 2t = 0$

iv.) $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$

[2 Marks]

b) i.) List advantages and disadvantages of using the explicit method in solving partial differential equations.

ii.) Use explicit method to solve the partial differential equation,

$$\frac{\partial^2 u(x,t)}{\partial x^2} - 2 \frac{\partial u(x,t)}{\partial t} = 0 \quad ; \text{ for } 0 \leq x \leq 2 \text{ and } 0 \leq t \leq 0.6$$

with the initial conditions,

$$u(x, 0) = f(x) = x(2 - x)$$

and the boundary conditions,

$$u(0, t) = 0$$

$$u(2, t) = t^2.$$

Use, $h = 0.5$ and $k = 0.2$, where h and k are step sizes along x and t axes respectively.

[7 Marks]

c) Briefly explain the procedure of finite difference solution technique for solving

Laplace equation $\left(\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0 \right)$ highlighting any differences with the solution technique used in part (b).

[3 Marks]